On the determination of the positions of points and planes after rotation through a definite angle about a known axis.

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I.

Rectangular axes.

TAKE three axes of reference OX, OY, OZ mutually at right angles.

A convention must be made as to the positive direction of rotation about any line, and that usually adopted is as follows. A positive rotation about OX rotates OY towards OZ, one about OY rotates OZ towards OX, and one about OZ rotates OX towards OY. To find the positive direction of rotation for any line ON we must suppose ON moved to coincide with one of the axes, say OZ, and then apply the preceding rule.

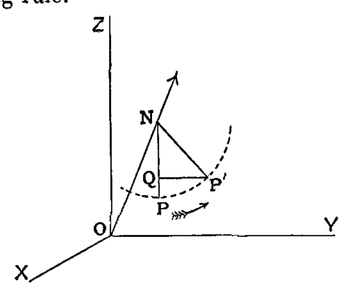

Fig. 1.

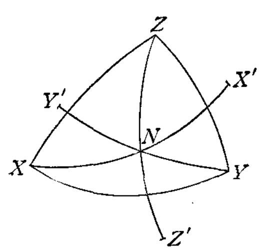

Fig. 2.

Taking the axes as indicated in the figure, in which OX is supposed drawn towards the observer, a positive rotation about any line ON together with a translation along ON produces right handed screwing. If the axes OX and OY were interchanged, the screwing would be left-handed.

The figure represents the displacement of a point P to the position P' by rotation through an angle ϕ about ON in the positive direction. The problem is to determine the co-ordinates $(x'y'z')$ of P' in terms of those of $P(xyz)$, ϕ and the given direction ON.

Let $\angle NOX = \lambda$, $\angle NOY = \mu$, $\angle NOZ = \nu$.

PN, $P'N$ are perpendicular to ON and $\angle PNP' = \phi$.

$P'Q$ is perpendicular to NP.

To find ON we notice that it is the projection of OP on ON. Hence

$$ON = OP \cos PON = OP\left\{\frac{x}{OP}\cos\lambda + \frac{y}{OP}\cos\mu + \frac{z}{OP}\cos\nu\right\}$$
$$= x\cos\lambda + y\cos\mu + z\cos\nu \;\ldots\ldots\ldots\ldots\; (1)$$

It is convenient to use the co-ordinates of N which are $\xi\eta\zeta$ where

$$\xi = ON\cos\lambda = (x\cos\lambda + y\cos\mu + z\cos\nu)\cos\lambda$$
$$\eta = ON\cos\mu = (x\cos\lambda + y\cos\mu + z\cos\nu)\cos\mu$$
$$\zeta = ON\cos\nu = (x\cos\lambda + y\cos\mu + z\cos\nu)\cos\nu$$

The direction of QP' is required.

The direction-cosines of ON are $\cos\lambda$, $\cos\mu$, $\cos\nu$.

$$,,\qquad NP\;,,\quad \frac{x-\xi}{NP},\ \frac{y-\eta}{NP},\ \frac{z-\zeta}{NP}.$$
$$\text{or say}\quad p\ ,\quad q\ ,\quad r\ .$$

Hence those of QP', which is perpendicular to both, are

$$\pm(r\cos\mu - q\cos\nu),\ \pm(p\cos\nu - r\cos\lambda),\ \pm(q\cos\lambda - p\cos\mu).$$

The ambiguous sign is determined by the fact that, if ON be moved to coincide with OZ and NP to the direction OX, QP' will then be in the direction OY. That is, when $\cos\lambda = 0 = \cos\mu$, $\cos\nu = 1$, $p = 1$, $q = 0 = r$, we must have

$$\pm(p\cos\nu - r\cos\lambda) = 1.$$

Hence the upper sign must be taken.

We are now in a position to write down the co-ordinates of P', because we know the lengths and directions of ON, NQ, QP'.

Thus $$x' = ON\cos\lambda + NQ.p + QP'(r\cos\mu - q\cos\nu)$$
$$= \xi + NP'\cos\phi.\frac{x-\xi}{NP} + NP'\sin\phi\,(r\cos\mu - q\cos\nu)$$
$$= \xi + \cos\phi\,(x-\xi) + \sin\phi\,(z\cos\mu - y\cos\nu)$$

$$\left.\begin{aligned} &\text{Or} & x'-x &= (1-\cos\phi)(\xi - x) + \sin\phi\,(z\cos\mu - y\cos\nu)\\ &\text{Similarly,} & y'-y &= (1-\cos\phi)(\eta - y) + \sin\phi\,(x\cos\nu - z\cos\lambda)\\ &\text{and} & z'-z &= (1-\cos\phi)(\zeta - z) + \sin\phi\,(y\cos\lambda - x\cos\mu)\end{aligned}\right\}\ldots\ldots (2);$$

wherein ξ, η, ζ have the values found above.

The relations (2), which give the new position of any point, are established in Minchin's *Statics*, II, p. 103, 1889; and his method of proof is that followed above.

If $x\ y\ z$ are required in terms of $x'y'z'$, it is necessary only to interchange x and x', &c. in the above formulæ and to change the sign of ϕ.

Suppose that it is required to find the new position of a plane

$$hx/a+ky/b+lz/c=1.$$

Now the point h/a, k/b, l/c is the pole P of this plane with respect to the sphere

$$x^2+y^2+z^2=1;$$

and the relation of pole and polar is unchanged by any rotation about 0. Hence the new position of the plane is

$$h'x/a+k'y/b+l'x/c=1;$$

where h'/a, k'/b, l'/c are the co-ordinates of the new position P' of the original pole P. They are therefore obtained from (2), and are

$$\left.\begin{aligned}
\frac{h'}{a}&=\frac{h}{a}\cos\phi+ON\cos\lambda\,(1-\cos\phi)+\sin\phi\left(\frac{l}{c}\cos\mu-\frac{k}{b}\cos\nu\right)\\
\frac{k'}{b}&=\frac{k}{b}\cos\phi+ON\cos\mu\,(1-\cos\phi)+\sin\phi\left(\frac{h}{a}\cos\nu-\frac{l}{c}\cos\lambda\right)\\
\frac{l'}{c}&=\frac{l}{c}\cos\phi+ON\cos\nu\,(1-\cos\phi)+\sin\phi\left(\frac{k}{b}\cos\lambda-\frac{h}{a}\cos\mu\right)
\end{aligned}\right\}\ldots.(3),$$

where $ON=\frac{h}{a}\cos\lambda+\frac{k}{b}\cos\mu+\frac{l}{c}\cos\nu$.

II. *Oblique axes.*

Choose rectangular axes as in case I, and let the oblique axes, referred to them, have direction cosines $(l_1m_1n_1)$, $(l_2m_2n_2)$, $(l_3m_3n_3)$.

Let OP make angles $\theta_1\,\theta_2\,\theta_3$ with the oblique axes.

$$\text{Then}\ \cos\theta_i=(l_ix+m_iy+n_iz)\div OP,\ (i=1,2,3).$$

It is required to find the angles $\theta_1'\ \theta_2'\ \theta_3'$ made by OP' with the oblique axes.

We have $\cos\theta_i'=(l_ix'+m_iy'+n_iz')\div OP'$, $(i=1,2,3)$.

Hence, multiplying the formulæ (2) respectively by $\frac{l}{OP}$, $\frac{m}{OP}$, $\frac{n}{OP}$ (dropping the suffix) and adding, we obtain

$$\cos\theta'=\cos\phi\cos\theta+(1-\cos\phi)\frac{l\xi+m\eta+n\zeta}{OP}+\sin\phi\begin{vmatrix} l & m & n\\ \cos\lambda & \cos\mu & \cos\nu\\ \frac{x}{OP} & \frac{y}{OP} & \frac{z}{OP}\end{vmatrix}$$

Now, if the axis of rotation ON makes angles $\omega_1\ \omega_2\ \omega_3$ with the oblique axes, then

$$\cos\omega_i = l_i\frac{\xi}{ON} + m_i\frac{\eta}{ON} + n_i\frac{\zeta}{ON}, (i=1, 2, 3).$$

$\therefore\ \cos\theta_i' = \cos\phi\cos\theta_i + (1-\cos\phi).\ \cos PON.\ \cos\omega_i + \sin\phi\,\Delta_i$......(4).
where

$$\Delta_i^2 = \begin{vmatrix} l_i^2+m_i^2+n_i^2, & l_i\cos\lambda+m_i\cos\mu+n_i\cos\nu, & l_i\frac{x}{OP}+m_i\frac{y}{OP}+n_i\frac{z}{OP} \\ l_i\cos\lambda+m_i\cos\mu+n_i\cos\nu, & \cos^2\lambda+\cos^2\mu+\cos^2\nu, & \cos\lambda\frac{x}{OP}+\cos\mu\frac{y}{OP}+\cos\nu\frac{z}{OP} \\ l_i\frac{x}{OP}+m_i\frac{y}{OP}+n_i\frac{z}{OP}, & \cos\lambda\frac{x}{OP}+\cos\mu\frac{y}{OP}+\cos\nu\frac{z}{OP}, & \frac{x^2+y^2+z^2}{O^2P} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \cos\omega_i & \cos\theta_i \\ \cos\omega_i & 1 & \cos PON \\ \cos\theta_i & \cos PON & 1 \end{vmatrix} \text{..........(5).}$$

It remains only to calculate $\cos PON$ in terms of $\omega_1\ \omega_2\ \omega_3$, $\theta_1\ \theta_2\ \theta_3$, and the angles α, β, γ between the oblique axes.

Let e, f, g be the direction ratios of OP.

Construct a parallelepiped, having OP as diagonal, and edges parallel to the oblique axes, and project on ON, we then obtain

$$\cos PON = e\cos\omega_1 + f\cos\omega_2 + g\cos\omega_3.$$

Again, projecting on the oblique axes, we get—

$$\cos\theta_1 = e + f\cos\gamma + g\cos\beta$$
$$\cos\theta_2 = e\cos\gamma + f + g\cos\alpha$$
$$\cos\theta_3 = e\cos\beta + f\cos\alpha + g$$

Eliminate e, f, g.

$$\therefore\ \begin{vmatrix} \cos PON & \cos\omega_1 & \cos\omega_2 & \cos\omega_3 \\ \cos\theta_1 & 1 & \cos\gamma & \cos\beta \\ \cos\theta_2 & \cos\gamma & 1 & \cos\alpha \\ \cos\theta_3 & \cos\beta & \cos\alpha & 1 \end{vmatrix} = 0$$

Or

$$\cos PON \begin{vmatrix} 1 & \cos\gamma & \cos\beta \\ \cos\gamma & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{vmatrix} + \begin{vmatrix} 1 & \cos\gamma & \cos\beta & \cos\theta_1 \\ \cos\gamma & 1 & \cos\alpha & \cos\theta_2 \\ \cos\beta & \cos\alpha & 1 & \cos\theta_3 \\ \cos\omega_1 & \cos\omega_2 & \cos\omega_3 & 0 \end{vmatrix} = 0 \text{......(6)}$$

Thus $\cos PON$ is found, and thence Δ_1, Δ_2, Δ_3, and the result may be stated as follows:

If the line OP, making angles $\theta_1\ \theta_2\ \theta_3$ with the axes, is turned through an angle ϕ about ON, making angles $\omega_1\ \omega_2\ \omega_3$ with the axes, into the position OP' having the direction-angles $\theta_1'\ \theta_2'\ \theta_3'$, then

$$\cos\theta_i' = \cos\phi\cos\theta_i + (1-\cos\phi)\cos PON\cos\omega_i + \sin\phi\,\Delta_i, (i=1, 2, 3).$$

To find the new position of any plane we notice that the direction cosines of the normal to the plane $h\frac{x}{a}+k\frac{y}{b}+l\frac{z}{c}=1$ are $\rho\frac{h}{a}, \rho\frac{k}{b}, \rho\frac{l}{c}$, where ρ is the perpendicular from O on the plane, and is unaltered by the rotation.

Thus the formulæ required are:

$$\left.\begin{aligned}\frac{h'}{a} &= \frac{h}{a}\cos\phi+(1-\cos\phi)\frac{\cos PON}{\rho}\cos\omega_1+\sin\phi\, D_1\\ \frac{k'}{b} &= \frac{k}{b}\cos\phi+(1-\cos\phi)\frac{\cos PON}{\rho}\cos\omega_2+\sin\phi\, D_2\\ \frac{l'}{c} &= \frac{l}{c}\cos\phi+(1-\cos\phi)\frac{\cos PON}{\rho}\cos\omega_3+\sin\phi\, D_3\end{aligned}\right\}\ldots\ldots(7)$$

where $$-\frac{\cos PON}{\rho} =$$

$$\begin{vmatrix} 1 & \cos\gamma & \cos\beta & h/a \\ \cos\gamma & 1 & \cos\alpha & k/b \\ \cos\beta & \cos\alpha & 1 & l/c \\ \cos\omega_1 & \cos\omega_2 & \cos\omega_3 & 0 \end{vmatrix} \div \begin{vmatrix} 1 & \cos\gamma & \cos\beta \\ \cos\gamma & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{vmatrix};\ldots\ldots(8)$$

$$D_1^2 = \begin{vmatrix} 1 & \cos\omega_1 & h/a \\ \cos\omega_1 & 1 & \rho^{-1}\cos PON \\ h/a & \rho^{-1}\cos PON & \rho^{-2} \end{vmatrix};\ \&c.;\ \ldots\ldots(9)$$

and ρ is given by

$$\begin{vmatrix} 1 & \cos\gamma & \cos\beta & h/a \\ \cos\gamma & 1 & \cos\alpha & k/b \\ \cos\beta & \cos\alpha & 1 & l/c \\ h/a & k/b & l/c & \rho^{-2} \end{vmatrix} = 0 \ldots\ldots\ldots\ldots(10)$$

When the angle of rotation is 180° the formulæ of transformation can be obtained directly as follows:

Let as before the plane $hx/a+ky/b+lz/c=1$ become

$$h'x/a+k'y/b+l'z/c=1.$$

The relative motion of the plane and axes is the same as if the plane were fixed and the axes were turned through 180° about ON. Let the axes meet a sphere with radius ON in X, Y, Z, Fig. 2, p. 343; and let their positions after a semi-revolution about ON be given by X', Y', Z'.

Then $$X'N=NX;\ Y'N=NY;\ Z'N=NZ.$$

And

$$\left.\begin{aligned}&\text{Cos } X'X=\cos 2\,XN=2\cos^2 XN-1=2\cos^2\omega_1-1.\\&\text{Cos } X'Y+\cos XY=2\cos XN\cos YN;\ \therefore \cos X'Y=2\cos\omega_1\cos\omega_2-\cos\gamma.\\&\text{Cos } X'Z+\cos XZ=2\cos XN\cos ZN;\ \therefore \cos X'Z=2\cos\omega_1\cos\omega_3-\cos\beta.\end{aligned}\right\}\ldots(11)$$

But $x\ y\ z$ being the point at distance r on OX' at which the plane is met, then

$$\left.\begin{aligned}r\cos XX'&=x+y\cos\gamma+z\cos\beta,\\r\cos YX'&=x\cos\gamma+y+z\cos\alpha,\\r\cos ZX'&=x\cos\beta+y\cos\alpha+z.\\1&=hx/a+ky/b+lz/c\end{aligned}\right\}\ldots\ldots\ldots(12)$$

Eliminating x, y and z, and replacing $\cos XX'$, &c. by their values given in (11), and r by its equivalent $a\div h'$, we have—

$$\begin{vmatrix}2\cos^2\omega_1-1 & 1 & \cos\gamma & \cos\beta\\ 2\cos\omega_1\cos\omega_2-\cos\gamma & \cos\gamma & 1 & \cos\alpha\\ 2\cos\omega_1\cos\omega_3-\cos\beta & \cos\beta & \cos\alpha & 1\\ \dfrac{(h'+h)-h}{a}, & \dfrac{h}{a} & \dfrac{k}{b} & \dfrac{l}{c}\end{vmatrix}=0.$$

This reduces to

$$\begin{vmatrix}\cos\omega_1 & 1 & \cos\gamma & \cos\beta\\ \cos\omega_2 & \cos\gamma & 1 & \cos\alpha\\ \cos\omega_3 & \cos\beta & \cos\alpha & 1\\ \dfrac{h_1+h}{2a\cos\omega_1} & \dfrac{h}{a} & \dfrac{k}{b} & \dfrac{l}{c}\end{vmatrix}=0\ldots\ldots(13);$$

which may be expressed by

$$(h_1+h)D-2a\cos\omega_1\Delta=0.$$

The formulæ of transformation are therefore

$$\frac{h'+h}{a\cos\omega_1}=\frac{k'+k}{b\cos\omega_2}=\frac{l'+l}{c\cos\omega_3}=\frac{2\Delta}{D}\ldots\ldots(14);$$

where

$$\Delta=\begin{vmatrix}\cos\omega_1 & 1 & \cos\gamma & \cos\beta\\ \cos\omega_2 & \cos\gamma & 1 & \cos\alpha\\ \cos\omega_3 & \cos\beta & \cos\alpha & 1\\ 0 & \dfrac{h}{a} & \dfrac{k}{b} & \dfrac{l}{c}\end{vmatrix},\ \text{and } D=\begin{vmatrix}1 & \cos\gamma & \cos\beta\\ \cos\gamma & 1 & \cos\alpha\\ \cos\beta & \cos\alpha & 1\end{vmatrix}.$$