Some applications of the Gnomonic Projection to Crystallography.

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1. LET the crystallographic axes OA, OB, OC of a crystal, whose axial ratios are $a:b:c$, meet a plane p in A, B, C. Let a plane through O, parallel to a crystal-face whose indices are h, k, l,

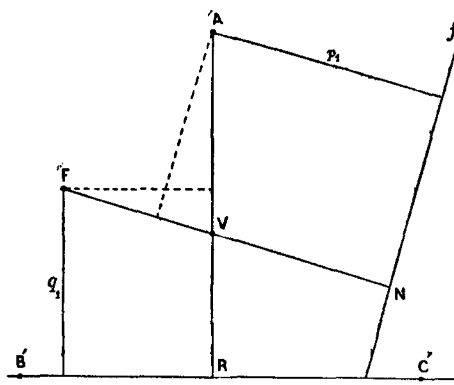

Fig. 1.

meet p in the line f (fig. 1); and let the line OF, perpendicular to the face, meet p at the point F. Let the perpendicular from O on p meet it in V, and let $OV = m$. Let

$$VOA = a,\ VOB = \beta,\ VOC = \gamma,\ FOA = \lambda,\ FOB = \mu,\ FOC = \nu,$$
$$VOF = \theta.$$

Then $$h:k:l = a\cos\lambda : b\cos\mu : c\cos\nu.$$

Let FV cut f in N; then FN is perpendicular to f and $FV \cdot VN = m^2$.

Let the perpendiculars from A, B, C on f be p_1, p_2, p_3. Then

$$p_1 = VN + VA\cos AVF.$$

But $$VN = m^2 \div FV = m\cot\theta,\quad VA = m\tan a,$$
$$\cos AVF = (\cos\lambda - \cos a \,.\, \cos\theta)\operatorname{cosec} a \,.\, \operatorname{cosec}\theta,$$

by spherical trigonometry.

$$\therefore \quad p_1 = m \cos\lambda \,.\, \sec a \,.\, \operatorname{cosec}\theta,$$

and $$\therefore \quad p_1 a \cos a : p_2 b \cos\beta : p_3 c \cos\gamma = h : k : l.$$

Hence, if we take ABC as triangle of reference in the plane p and suitable triliteral coordinates (xyz), the equation of the line f is

$$hx + ky + lz = 0.$$

The indices of the edge in which the faces (hkl) and $(h'k'l')$ intersect are the same as the coordinates of the intersection of the lines whose triliteral equations are

$$hx + ky + lz = 0 \quad \text{and} \quad h'x + k'y + l'z = 0.$$

Therefore the line through O parallel to the edge $[HKL]$ meets p in the point (HKL).

Let the perpendiculars from O to the faces (100), (010), (001) meet p in $A'B'C'$. Then F is the pole of f, A of $B'C'$, B of $C'A'$, C of $A'B'$ with respect to a circle of radius $\sqrt{-m^2}$ whose centre is V[1]. Let q_1, q_2, q_3 be the perpendiculars from F on the sides of the triangle $A'B'C'$. Produce AV to cut $B'C'$ in R. Then

$$q_1 = VR + VF \cos AVF = m \cos\lambda \,.\, \operatorname{cosec} a \,.\, \sec\theta$$

as before. Therefore

$$q_1 a \sin a : q_2 b \sin\beta : q_3 c \sin\gamma = h : k : l.$$

Hence if we form the gnomonic projection on p of the poles and zones of the crystal which lie on a sphere of radius m whose centre is V, and if we choose a suitable triangle of reference on p and suitable triliteral coordinates, then the point (hkl) is the projection of the pole of the face (hkl) and $Hx + Ky + Lz = 0$ is the projection of the zone $[HKL]$.

The gnomonic projection of the poles and zones on p is the reciprocal polar with respect to a circle of radius $\sqrt{-m^2}$, whose centre is V, of the figure formed by drawing through O planes parallel to the faces cutting p in lines, and lines parallel to the edges cutting p in points.

2. We shall now take illustrations of the use of these theorems.

The anharmonic ratio of four cozonal faces $(h_1k_1l_1)$, $(h_2k_2l_2)$, $(h_3k_3l_3)$, $(h_4k_4l_4)$ is evidently equal to the anharmonic ratio of the range formed by the gnomonic projection of their poles. Now these four poles are the points $(h_1k_1l_1)$, &c., the triangle of reference being suitably chosen, and therefore the anharmonic ratio of the faces is equal to the anharmonic

[1] Since in a plane triangles reciprocal with respect to a circle are homologous, we have incidentally proved that the arcs joining corresponding vertices of a spherical triangle and its polar triangle are concurrent, and that the intersections of corresponding sides of the two triangles lie on a great circle.

ratio of the pencil formed by the lines joining these points to A', i. e., by the lines $yl_1 = zk_1$, &c. Therefore the anharmonic ratio of the four faces

$$= \left(\frac{k_3}{l_3} - \frac{k_1}{l_1}\right)\left(\frac{k_4}{l_4} - \frac{k_2}{l_2}\right) \div \left(\frac{k_4}{l_4} - \frac{k_1}{l_1}\right)\left(\frac{k_3}{l_3} - \frac{k_2}{l_2}\right)$$

$$= (k_1l_3 - k_3l_1)(k_2l_4 - k_4l_2) \div (k_1l_4 - k_4l_1)(k_2l_3 - k_3l_2).$$

Again project on to a plane perpendicular to an n-al rotation-axis of the crystal. Let A_1 be the projection of the pole of a face; then $A_2, A_3, A_4, \ldots, A_n$ are also projections of poles, $A_1A_2 \ldots A_n$ being a

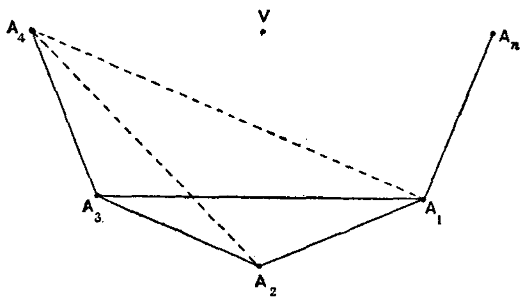

Fig. 2.

regular n-sided polygon (fig. 2). Take $A_1A_2A_3$ as triangle of reference. Then the areal coordinates of A_4 and A_n are

$$\frac{A_2A_3A_4}{A_1A_2A_3}, \frac{A_3A_1A_4}{A_1A_2A_3}, \frac{A_1A_2A_4}{A_1A_2A_3} \quad \text{and} \quad \frac{A_2A_3A_n}{A_1A_2A_3}, \frac{A_3A_1A_n}{A_1A_2A_3}, \frac{A_1A_2A_n}{A_1A_2A_3}.$$

But $A_3A_1A_4 = A_3A_1A_n$,

and $$\therefore \frac{A_2A_3A_n}{A_2A_3A_4} = \frac{A_1A_2A_4}{A_2A_3A_4} = \frac{A_1A_4}{A_2A_3} = 1 + 2\cos\frac{2\pi}{n}$$

is rational. Therefore $n = 2, 3, 4$, or 6.

This proof may be readily extended to the case of symmetry-axes of the second sort (axes of alternating symmetry). It assumes neither the rationality of the anharmonic ratio of four cozonal faces nor the fact that the symmetry-axis is parallel to a possible edge.

3. Suppose we have n points in a plane, and suppose every pair of points joined by a straight line. Suppose that one of these lines passes through r of the points $P_1, P_2, \ldots, P_{r-1}, P_r$. Now if one of these points (P_r say) is moved out of the line, the total number of lines is increased by $(r-1)$; for the r lines $P_1P_r, P_2P_r, \ldots, P_{r-1}P_r, P_1 \ldots P_{r-1}$ replace the

single line $P_1 \dots P_r$. If now P_{r-1} is also moved out of the line, the total number of lines is increased by $(r-2)$ more lines, and so on. Suppose this process repeated until no line passes through more than two points; the number of lines is now $\frac{1}{2}n(n-1)$. Let u_r be the number of lines passing through r of the points and no more. Then we have

$$u_2 + u_3 + \dots + u_n = \tfrac{1}{2}n(n-1) - \sum_2^n \{(2+3+\dots+\overline{r-1})\,u_r\},$$

and therefore

$$n(n-1) = \sum_2^n r(r-1)\,u_r \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$[1].

Again, in the above let v_r be the number of points at which r lines intersect, so that

$$\Sigma v = n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}.$$

Then the total number of lines would be

$$v_1 + 2v_2 + \dots + (n-1)\,v_{n-1},$$

if we reckoned each line through r points as equivalent to r lines. Therefore

$$v_1 + 2v_2 + \dots + (n-1)\,v_{n-1} = 2u_2 + 3u_3 + \dots + nu_n,$$

or

$$\sum_2^n \{(r-1)\,v_{r-1} - ru_r\} = 0 \, . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}.$$

Combining (i), (ii), and (iii) we have

$$n = \sum_2^n r\,(v_{r-1} - u_r) \quad \text{and} \quad n(n-2) = \sum_2^n r\,(ru_r - v_{r-1}).$$

Suppose we have n lines in a plane, and suppose that every pair of lines intersects in a point. Let U_r be the number of points lying on r of the lines, and V_r the number of lines on which r points lie. Then by reciprocation

$$n(n-1) = \sum_2^n r(r-1)\,U_r, \quad \sum_2^n \{(r-1)V_{r-1} - rU_r\} = 0, \text{ \&c.}$$

Using the gnomonic projection we deduce the following:—

If on a crystal with n faces (two parallel faces being considered equivalent to only a single face) there are u_r zones containing r faces and v_r faces lying in r zones ($\Sigma v = n$), then

$$n(n-1) = \sum_2^n r(r-1)\,u_r, \quad \sum_2^n \{(r-1)\,v_{r-1} - ru_r\} = 0,$$

$$\sum_2^n r\,(v_{r-1} - u_r), \quad \text{and} \quad n(n-2) = \sum_2^n r\,(ru_r - v_{r-1}).$$

[1] Another proof is given by E. von Fedorow, Abhandl. k. bayer. Akad. Wiss. München, Math.-phys. Cl., 1900, vol. xx, p. 496.

Summary.

1. In the gnomonic projection of the poles of a crystal on any plane the projection of the pole of (hkl) is the point (hkl), and the projection of the zone $[HKL]$ is $Hx+Ky+Lz=0$, a suitable triangle of reference and suitable triliteral coordinates being taken.

2. By means of this theorem simple proofs of well-known crystallographic theorems may be obtained.

3. Some relations between the number (u_r) of zones of a crystal containing r faces, and the number of faces (v_r) lying in r zones may be found; e.g.,

$$n(n-1)=\sum_2^n r(r-1)u_r, \quad \sum_2^n \{(r-1)v_{r-1}-ru_r\}=0,$$

where $$\Sigma v_r = n.$$

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