The Construction of Crystallographic Projections.

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§ 1. It is proposed in this paper to describe some simple graphical constructions of theoretical or practical interest. They may be easily performed with ruler and compasses, though the stereographic and gnomonic nets[1] will often save labour. To economize space the proofs of the constructions are left to the reader.

§ 2. First take the case of the gnomonic projection. Given the projections A, B, C, Q (fig. 1) of the poles of the faces (100), (010), (001), (111); find the projection of the pole of any other face (hkl).

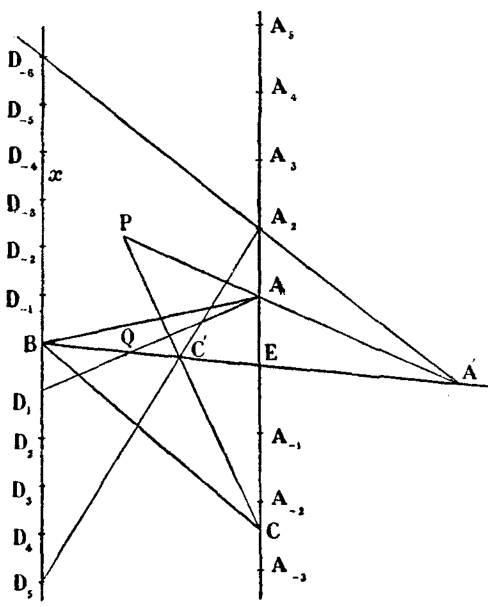

Fig. 1.

Through B draw a line x parallel to AC. Let AQ meet x in D_1, and let BQ meet CA in E. Divide x into equal intervals . . ., $D_{-2}D_{-1}$, $D_{-1}B$, BD_1, D_1D_2, D_2D_3, . . . and CA into equal intervals . . ., $A_{-2}A_{-1}$, $A_{-1}E$, EA, AA_2, A_2A_3, ... Plot the intersections A', C' of BE with A_kD_l, A_kD_h respectively, and the intersection P of AA', CC' (A_1 coincides with A and D_0 with B). P is the point required; for it may readily be shown that the triliteral coordinates of P are (h, k, l), if the coordinates of Q are $(1, 1, 1)$[2]. In the diagram, P is the projection of the pole of $(52\bar{6})$.

[1] See Zeits. Kryst. Min., 1902, vol. xxxvi, p. 14; Min. Mag., 1904, vol. xiv, p. 18.
[2] See p. 104.

The construction holds if h or $l=0$, but not if $k=0$. If $k=0$, we may draw a line through C parallel to AB and proceed similarly; or we may find the projection R of the pole of $(hk'l)$, where $k'\neq 0$, and produce BR to cut CA in the required point.

If Q were the projection of the pole of (uvw) instead of (111), we should take A_{wuk} instead of A_k, D_{uvl} instead of D_l, and D_{vwh} instead of D_h.

More generally, if A, B, C, Q are the projections of the poles of $(h_1k_1l_1)$, $(h_2k_2l_2)$, $(h_3k_3l_3)$, $(h_4k_4l_4)$ instead of (100), (010), (001), (111), we replace h, k, l respectively by $M_2M_3\mu_1$, $M_3M_1\mu_2$, $M_1M_2\mu_3$; where M_1, M_2, M_3 are the cofactors of m_1, m_2, m_3 in the determinant $[h_1k_2l_3m_4]$, and μ_1, μ_2, μ_3 are the cofactors of m_1, m_2, m_3 in the determinant $[hk_1l_2m_3]$.

§ 3. Again, joining the points A, B, C, Q in every possible way we obtain the projections of zones meeting in points which are the projections of poles of crystal-faces. Joining these points we get the projections of more zones which meet in the projections of more poles; and, proceeding in this way, we get the projection of every possible pole. However, except for the poles of faces with very simple indices, this process is more tedious than the method of § 2.

§ 4. Suppose we wish to find the angle between two faces the projections of whose poles are P and Q.

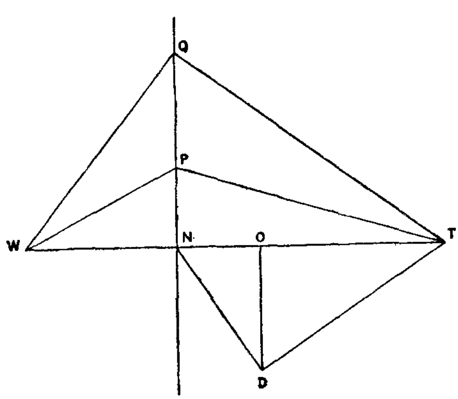

Fig. 2.

Let O (fig. 2) be the centre of the projection and r the radius of the sphere of reference. Draw ON perpendicular to PQ, and draw OD perpendicular to ON and equal to r. Take W on ON such that $NW=ND$.

Then PWQ is the angle required, since PQ subtends equal angles at W and at the centre of the sphere of reference.

Again, suppose we wish to find the angle between two zones whose projections are TP and TQ. Draw OD perpendicular to TO and equal to r. Draw DN perpendicular to TD. Draw NPQ perpendicular to TN, and take W on TN such that $NW=ND$. Then PWQ is the angle required.

This result is important, for it shows that the gnomonic projection of the poles of a crystal can be readily constructed when we know the angles between its faces and zones; and that, conversely, when we are given the gnomonic projection, we can measure graphically the angles.

The result can be also applied at once to the graphical solution of spherical triangles, &c.

§ 5. Suppose we are given the gnomonic projection P of any pole of a crystal, and we wish to find the projection of the corresponding pole

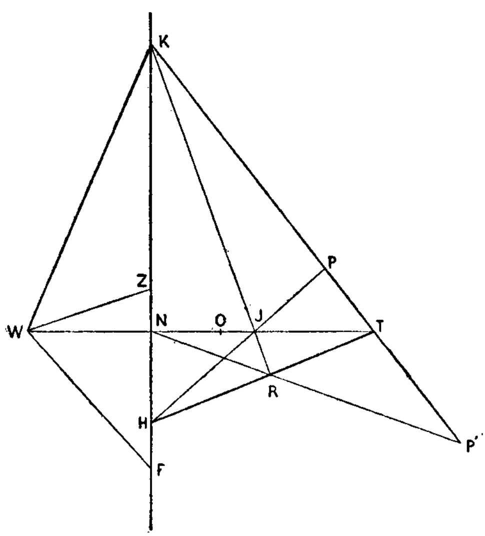

Fig. 3.

of a twin-crystal. Let the crystals be twinned about an axis meeting the plane of projection in T (fig. 3). Produce TO to N so that $TO \cdot ON = r^2$, and draw KNH perpendicular to TN meeting TP in K. Plot the intersection J of TN and HP, the intersection R of KJ and TH and the intersection P' of NR and PT. P' is the point required, since the range (PP', KT) is harmonic, and TK subtends a right angle at the centre of the sphere of reference.

The lines TH, TN, HK are drawn once for the whole crystal, and then only one line (TP) has to be drawn for each pole.

§ 6. Suppose now we wish to find the new projection of the pole when the crystal is rotated through an angle a (not$=180°$) about the axis. Take W on TN such that $WN^2=NO.NT$. Make the angles $KWZ=a$, $KWF=\frac{1}{2}(180°+a)$. Then the point required is the intersection of PF and TZ.

§ 7. We now take the case of the stereographic projection.

It is easy, when we are given the stereographic projection of the zones and poles of a crystal, to find the gnomonic projection on a parallel plane and vice versa. For if the stereographic projection of a zone meets the principal circle j of the projection (whose radius is r and centre O) in E and F, and the tangents at E and F to the projection of the zone meet at S, the gnomonic projection of the zone is the straight line through S perpendicular to OS.

§ 8. Suppose we are given the circle j and the stereographic projections A, B, C, Q of the poles of the faces (100), (010), (001), (111). Let EF be the diameter of j passing through the intersection of BC with the common chord of j and any circle through B and C. Then the circle $BCEF$ is the stereographic projection of the zone [100] [1]. We can then by § 7 find the gnomonic projection of the zone [100], and similarly of the zone $[0\bar{1}1]$. Let the straight lines OB, OC meet the former in B', C', and let OA, OQ meet the latter in A', Q'. Then A', B', C', Q' are the gnomonic projections of the poles of (100), (010), (001), (111) respectively. We can therefore find the gnomonic projection of any zone or pole by § 2 or § 3, and then we obtain the stereographic projection by § 7.

§ 9. An alternative to the method of § 8 is the following. Let the stereographic projections of the zones $[10\bar{1}]$ and $[\bar{1}10]$ meet the circle ABC in B_1 and C_1, and let the straight lines BB_1 and CC_1 meet in Q_1. Let the gnomonic projection of the zone $[HKL]$ for a crystal, such that A, B, C, Q are the gnomonic projections of the poles of (100), (010), (001), (111), meet the circle ABC in X and Y. Then the stereographic projection of the zone $[HKL]$ will pass through X and Y [2].

§ 10. Suppose we are given the stereographic projection P of any

[1] This construction solves the problem: 'Draw a circle through two given points bisecting the circumference of a given circle.'

[2] See Phil. Mag., 1905, ser. 6, vol. ix, p. 87.

pole of a crystal, and we wish to find the projection P' of the corresponding pole of a twin-crystal. Let T be the projection of the point in which the twinning-axis meets the sphere of reference (whose centre is O and radius is r as before). Draw WOU perpendicular to OT, such that $WO=OU=r$. Then, if the circles WPU, $WP'U$ meet OT in L and L', TW bisects the angle LWL': this gives us the position of the circle $WP'U$. Or, again, if J, S, J' are the centres of the circles WPU, WTU, $WP'U$, SW bisects the angle JWJ'.

The results of the preceding sections hold good in general if we substitute 'extremity of the zonal axis $[hkl]$' for 'pole of the face (hkl)' and 'great circle whose plane is parallel to the face (hkl)' for 'zone $[hkl]$.'

§ 11. We now show how to construct the orthographic projection on a plane Π of a crystal whose faces $f_1, f_2, f_3, \ldots$ are at distances $p_1, p_2, p_3, \ldots$ from the centre C of the crystal. We consider p_1 positive or negative according as the perpendicular from C on f_1 makes an acute or obtuse angle with the perpendicular CO from C on Π; and so for $p_2, p_3, \ldots$. Construct the gnomonic projection of the poles on Π. Let P_1, P_2, P_3, ... be the projections of the poles of the faces $f_1, f_2, f_3, \ldots$. Draw P_1Q_1 perpendicular to OP_1 and equal to r. On OQ_1 take $OR_1=p_1$, and draw R_1S_1 perpendicular to OQ_1 meeting OP_1 in S_1. Draw a line l_1 through S_1 perpendicular to OP_1. Construct lines l_2, l_3, ... similarly. Then the line through the intersection of l_1 and l_2 perpendicular to the line P_1P_2 is the orthographic projection of the edge in which the faces f_1 and f_2 meet. Similarly we get the orthographic projection of all the edges.

Summary.

In this paper graphical constructions are given for the solution of the following problems:—

Given the gnomonic or stereographic projections of the poles of four faces, find the projection of the pole of any other face.

Given the gnomonic or stereographic projections of the poles of a crystal, find the projections of the poles of a twin-crystal.

Given the projection of two zones, find the angle between them.

Given the gnomonic projection of the poles of a crystal, find the orthographic projections of its edges, &c.

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