Gnomonic Projections on two planes.

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THE gnomonic projection in which the faces of crystals are represented by the points where their normals through a fixed point—the centre of projection—meet a plane—the plane of projection—is already widely used by crystallographers. I propose in the following pages to show the advantages of the simultaneous use of two such planes of projection.

It is in most cases convenient for the two planes to be at right angles, and for the centre of projection to be at an equal (unit) distance from each. Circumstances, however, in some cases render it desirable to vary these conditions. When the planes are at right angles, one may conveniently be described as the horizontal plane, and the other as the vertical plane.

In the diagrams, the projections of face-normals, and other points, on the horizontal plane are denoted by capital letters, and those on the vertical plane by small letters. Projections of zone-planes, and other lines, on the horizontal plane are represented by continuous lines, or, if there be two successive positions of the horizontal plane, those on one by continuous and those on the other by discontinuous lines; similar projections, and other lines, on the vertical plane are represented by lines of dots. The word projection, except where otherwise stated, is used throughout this paper in the sense of gnomonic projection.

The centre of projection is always designated by C, the foot of the perpendicular from it on the horizontal and vertical planes by P and p respectively, and the base-line or line of intersection of the horizontal and vertical planes by XX'. The perpendiculars from P and p on the base-line meet it at the same point M, and are ordinarily of unit length (12·5 mm. in the accompanying figures).

(1) If the projection of a zone-plane on the horizontal plane be given,

the projection on the vertical plane can be obtained by the following construction :—

Let TU (fig. 1) be the projection on the horizontal plane and let it meet XX' in U. Draw PQ parallel to XX' and meeting TU in Q, and QR at right angles to XX' and meeting it in R. Then the line pR is parallel to the line CQ (not shown in the diagram) in which the given zone-plane is intersected by a plane through C parallel to the vertical plane. The same zone-plane will intersect the vertical plane in a parallel line, for the lines in which two parallel planes are intersected by a third plane are parallel to one another. If therefore a line Ut be drawn parallel to Rp it will represent the projection of the given zone-plane on the vertical plane.

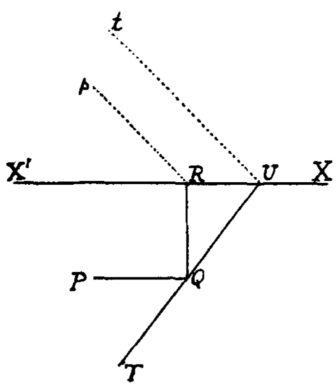

Fig. 1.

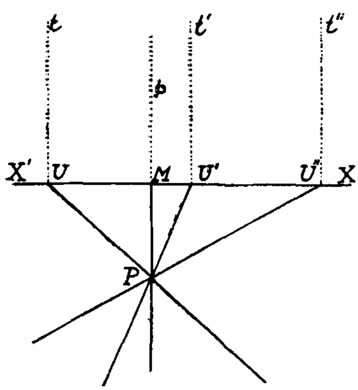

Fig. 1 a.

If the projection of a zone-plane on the horizontal plane pass through P (fig. 1 a), R will be at M on the straight line joining Pp, and the projection on the vertical plane will be at right angles to the base-line XX'. Vice versa, if the projection on the horizontal plane be at right angles to the base-line, the projection on the vertical plane will pass through the point p.

If the projection on the horizontal plane be parallel to the base-line the projection on the vertical plane will also be parallel to the base-line, and its position may be obtained by a projection on a subsidiary vertical plane at right angles to the first.

Let TU (fig. 2 or 2 a) be the projection of the given zone-plane, and let the base-line X_1X_1' of the subsidiary vertical plane meet TU in T and XX' in O. Join Tp_1 and produce it to meet XX' in t_1. Then

Ot_1 represents the distance from the horizontal plane of the intersection of the given zone-plane with the original vertical plane. On X_1X_1' make $Ot = Ot_1$, and draw tu parallel to XX', then tu is the projection of the given zone-plane on the vertical plane.

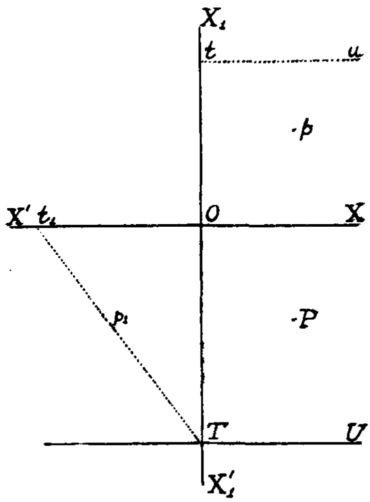

Fig. 2.

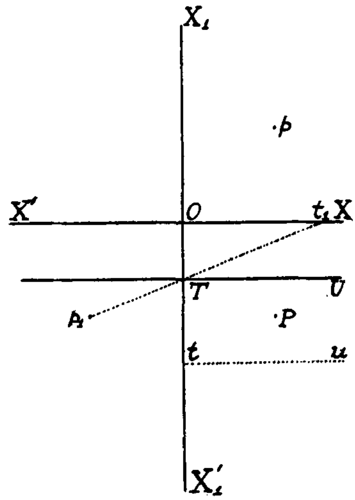

Fig. 2 a.

If the projection on the horizontal plane be parallel to the base-line and pass through P, the projection on the vertical plane will be entirely at infinity, and vice versa, if the projection on the vertical plane pass through p, that on the horizontal plane will be entirely at infinity.

(2) Given the projection of a face-normal on the horizontal plane, to find the projection on the vertical plane.

This is solved by taking the projections of any two convenient zone-planes meeting in the face-normal and finding their projections on the vertical plane. The point of intersection of these will be the projection of the required face-normal.

(3) The projection of zone-planes and face-normals on an inclined plane may be found by taking a vertical plane at right angles to the horizontal and inclined planes. The projections on the vertical plane are first found and thence those on the inclined plane. If the inclined plane be at the same distance from the centre of projection as the horizontal plane, it may be considered as a new horizontal plane obtained from the old horizontal plane by rotation about a horizontal axis through the centre of projection.

(4) Given the projection of a zone-plane $T_1 U_1$ on the horizontal plane, to find its projection on another plane obtained by rotating the first through an angle θ about a horizontal axis through the centre of projection.

Draw a vertical plane at right angles to the axis of rotation Cp, and let the base-line $X_1 X_1'$ meet $T_1 U_1$ at U (fig. 3). Draw $P_1 Q_1$ parallel, and $Q_1 R_1$ at right angles to $X_1 X_1'$. Then $U_1 t$, parallel to $R_1 p$, will be the projection of the zone-plane on the vertical plane. Draw the line pM_2 making an angle θ with $P_1 M_1 p$, and let $pM_2 = pM_1 =$ unity. Through M_2 draw $X_2 X_2'$ at right angles to pM_2, meeting pR_1 in R_2 and tU_1 in U_2. Draw $Q_2 R_2$ at right angles to $X_2 X_2'$ and equal to unity. Then $Q_2 U_2$ is the projection of the given zone-plane on the new horizontal plane.

For both the straight lines CQ_1 and CQ_2 pass through C and are parallel to pR_1; they are therefore the same straight line. Q_2 is therefore a point on the required projection on the new horizontal plane, and the same is the case with U_2.

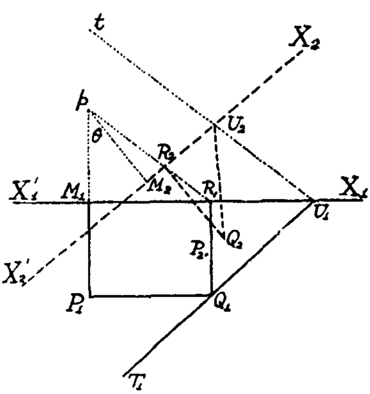

Fig. 3.

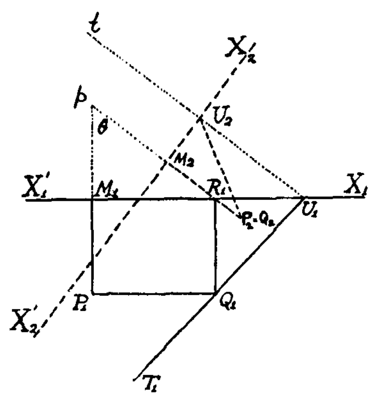

Fig. 3 *a*.

If the line pM_2 coincide with the line pR_1 the construction is somewhat simplified as shown by the accompanying diagram (fig. 3 *a*). The second horizontal plane will then be at right angles to the line CQ_1 and the point Q_2 will coincide with the point P_2, so that CP_2 is the line CQ_1 in a vertical position with regard to the new horizontal plane. $P_2 U_2$ is therefore the projection required.

(5) The position of the projection of a face-normal after rotation may be obtained by considering it as the point of intersection of the pro-

jections of two convenient zone-planes, and finding the positions of these, and consequently of their intersection, on the second horizontal plane.

(6) To find the angle between any two face-normals whose projections are given.

The horizontal plane is rotated round the projection of the common zone-plane till it passes through the centre of projection and the face-normals. The true angle between these will then be obtained. This is, in fact, the usual construction. Let U and U' (fig. 4) be the projections of the face-normals. Then C_2U and C_2U' are the positions of the face-normals in the horizontal plane after rotation, and the angle UC_2U' is the true angle between them.

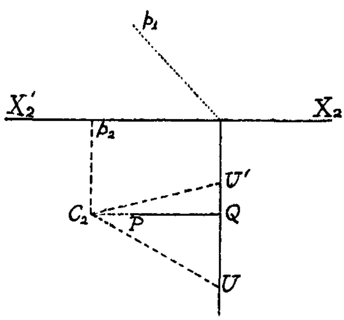

Fig. 4.

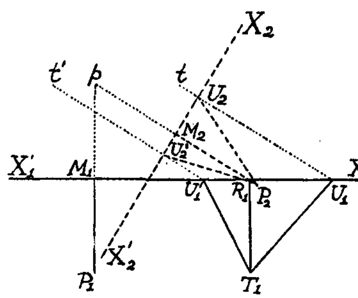

Fig. 5.

(7) Given the projection of a face-normal and a zone to which it belongs, to find the projection of another face-normal in the zone making a given angle with the first.

This is solved on the same principles as the previous problem.

(8) To find the angle between any two zone-planes whose projections are given.

Let the projections of the zone-planes be T_1U_1 and T_1U_1' (fig. 5) intersecting in T_1, the projection of their common face-normal. Assume a vertical plane with a base-line X_1X_1' parallel to the line P_1T_1, and intersecting the projections of the given zone-planes in U_1 and U_1'. Draw T_1R_1 at right angles to X_1X_1' intersecting it at R_1. Then U_1t and $U_1't'$, parallel to R_1p, will be the projections of the given zone-planes on the vertical plane.

Rotate the horizontal plane about Cp (the straight line through the centre of projection at right angles to the vertical plane) through the angle M_1pR_1 so that it may take up a position at right angles to the line CT_1. The new base-line will then be the line $X_2M_2X_2'$ drawn at right angles to pR_1 at unit distance pM_2 from p. Let it intersect U_1t and $U_1't'$ in U_2 and U_2'. In pR_1 make M_2P_2 equal to unity.

Then CP_2 is the same straight line as CT_1, and P_2U_2 and P_2U_2' are the projections of the given zone-planes on the new horizontal plane. They are therefore lines in which those zone-planes meet a plane at right angles to their line of intersection, and the plane angle between them is the angle between the zone-planes.

(9) Given the projection of a zone-plane and that of a face-normal in it, to find the projection of another zone-plane including this face-normal and making a given angle with the first zone-plane.

This is solved on the same principle as the last problem. The horizontal plane is tilted till it is at right angles to the common face-normal. The projection of the required zone-plane is drawn at the given angle, and the horizontal plane rotated back to its first position.

(10) By the help of these constructions it is possible to solve spherical triangles where two sides and the included angle are given, or two angles and the included side, or two sides and the angle opposite one of them. The sides are represented by zone-planes and the angular points by face-normals.

In the last case the face-normal corresponding to the given angle is brought into the position CP and the projections of the two adjoining sides are drawn. On the projection of the adjoining side whose dimensions are given a corresponding length[1] is measured off (No. 7), and in this way the position of the face-normal is obtained which corresponds to the angle between the two sides whose dimensions are given. This face-normal is now brought into the position CP and a circle described round its projection at a distance representing the length of the side opposite the known angle. The intersections of this circle with the projection of the side of unknown length give the two spherical triangles satisfying the required conditions.

(11) Given the projection of a face-normal, to find the locus of the projections of face-normals making a given angle with it.

Let T_1 (fig. 6) be the projection of the given face-normal. Rotate the horizontal plane so as to bring the face-normal CT_1 into a vertical position CP_2. Draw a circle round P_2 as centre at a distance corre-

[1] Which would be equal in this position to the tangent of the circular measure of the side.

sponding to the angle θ, and let D_2 and D_2' be the extremities of the diameter parallel to X_2X_2'. Find the projections D_1 and D_1' on the original horizontal plane of the face-normals of which D_2 and D_2' are the projections on the second horizontal plane, then D_1D_1' is the major axis of the ellipse that forms the required locus and E_1 is its centre. Two other points K_1 and K_1' on the ellipse can be found on the line through T_1 at right angles to P_1T_1 by the method of (7), and from these data the ellipse can easily be constructed by the help of the auxiliary circle $D_1N_1M_1D_1'$.[1]

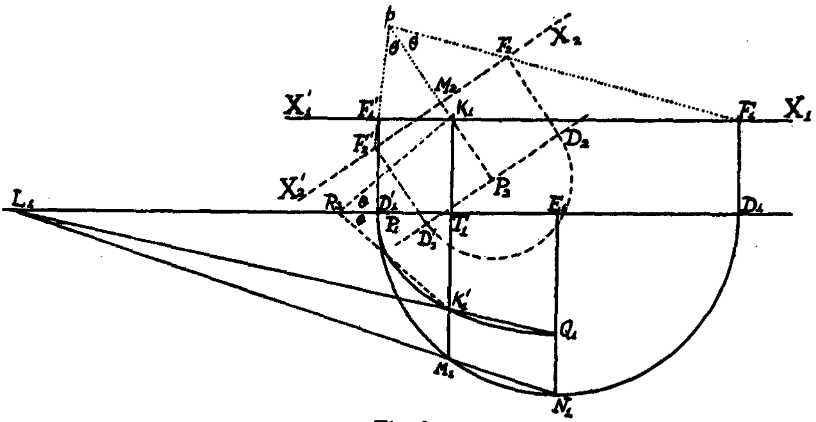

Fig. 6.

(12) By the help of this problem a spherical triangle can be solved in which the three sides are given. The other cases, where three angles or two angles and the side opposite one of them are given, can be solved by taking the polar triangle.

(13) Given the projection of a twin-axis and a zone-plane, to find the position of the corresponding twinned zone-plane.

The horizontal plane is rotated so as to bring the twinning axis into the position CP. The twinned position of the given zone-plane will be the parallel line on the opposite side of and at an equal distance from P. The horizontal plane can then be rotated back to its original position and the projection of the twinned zone-plane found in the usual manner.

[1] Let N_1 be any point on the circle, and M_1 the point on the circle where $K_1'T_1$ (produced) intersects it. Draw N_1M_1 meeting D_1D_1' in L_1. Draw L_1K_1' meeting at Q_1 the perpendicular from N_1 on D_1D_1'; then Q_1 is a point on the ellipse. In the diagram (fig. 6) Q_1 is the extremity of the minor axis of the ellipse.

(14) Given the projection of a twin-axis and that of a face-normal, to find the projection of the normal of the twinned face.

This can be done by taking two convenient zone-planes intersecting in the face-normal and determining their corresponding twinned zone-planes. The intersection of these projections will give that of the twinned face-normal.

All these constructions will apply to *linear projection*, if for zone-planes and face-normals there are substituted planes and lines parallel to the faces and edges of the crystal and passing through the centre of projection. The planes and lines of the linear projection are in fact the polar representatives of the face-normals and zone-planes of the gnomonic projection of the same crystal.

Both linear and gnomonic projection are of great convenience in the drawing of crystals, as they give at once the direction of the edges of the crystal in an orthogonal projection on the horizontal plane. In the gnomonic projection these are given by the lines drawn from P at right angles to the projection of the zone-plane: in the linear projection, by the lines joining P to the points representing the crystal-edges.

By rotating the horizontal plane through a convenient angle about any convenient axis it is easy to get a projection from which a drawing of the crystal may be obtained in a position which is adapted for showing its shape and symmetry.

In some cases there are other graphic solutions which involve fewer lines than those I have given in this paper, but the rationale is often not very obvious except to the expert mathematician, and to most mineralogists and students they will not afford so much insight into the spacial relations of the faces and zones of a crystal. In actual practice, too, some of the lines shown in the diagrams illustrating this paper may be omitted, and by judicious procedure the same lines are available for several different constructions. When two or three zones have been dealt with, the remainder can be worked out from them with only occasional resort to three dimensional methods.

I am indebted to Mr. T. W. Tyrrell for carefully executing the drawings illustrating this paper.