On some twins of calcite; and on a simple method of drawing crystals of calcite and other rhombohedral crystals, and of deducing the relations of their symbols.

By W. J. Lewis, M.A.
Professor of Mineralogy in the University of Cambridge.

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1. Having had occasion recently to draw the twins of calcite, I noticed the advantage of the twin-axes being all placed in the plane of the paper. The representation of the rhombohedron which results from this condition is so unsatisfactory that, obvious as the idea must have been to all crystal draughtsmen, few, if any, seem to have tested the utility of the projection. I desire to point out a few of its advantages.

2. *The rhombohedron.*

Fig. 1 shows a rhombohedron in two positions—the upper one being a plan on the central horizontal (equatorial) plane, and the lower one an elevation with the principal axis VV', the polar edge $V\mu$, and the polar diagonal $V'\mu$ all in the paper. All rhombohedra are projected in the same way: all that need be given is the angle $VV'\mu$, which is the complement of the angle made by a face with the base. Thus, for the fundamental rhombohedron of calcite, shown in fig. 1 and by interrupted lines in figs. 2 and 3, the angle $VV'\mu$ is 45° 23·4′.

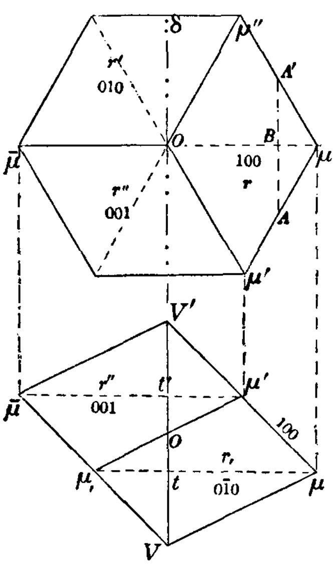

Fig. 1.

The figure is easily drawn. A vertical line being drawn for the principal axis, three equal lengths, $V't'$, $t't$, and tV, are marked off on it. Through V' a second line is drawn at the known angle $VV'\mu$, meeting horizontal lines through t' and t at the points μ', μ. Then clearly $V'\mu' = \mu'\mu$; and

$\mu t = 2\mu' t' = 2\mu_{,} t$. The figure is then completed by drawing the large parallelogram $V'\mu V$, and bisecting it by the median edge $\mu'\mu_{,}$. The points A, B, A' in the plan are projected in the same point in the elevation—midway between μ and μ'. They are important points in the geometry of rhombohedral crystals, and the distinction must be borne in mind.

From the method of drawing, it is clear that the median edge of the two faces shown is in the case of every rhombohedron parallel to the paper, and is projected in a line through the centre, which may be taken as origin. After drawing the fundamental rhombohedron it is advantageous to draw through the median coigns μ faint vertical lines, for the similar coigns of every derived rhombohedron will lie on them, if it is drawn to the same horizontal scale.

The figure being taken to represent the fundamental rhombohedron $\{100\}$ of Miller$=R$ of Naumann, the Millerian axis OX coincides with with $O\mu'$, and the parameter on it may be taken equal to $V\mu$; whilst the axes OY and OZ and the parameters on them are all projected in a line parallel, and equal, to $V\mu_{,}$. The face (100) is projected in the diagonal $V'\mu$; the upper parallelogram is (001), and the lower $(0\bar{1}0)$.

Corollary 1. Now, $\tan V'OX = \tan V'V\mu = \dfrac{\mu t}{Vt} = \dfrac{2\mu_{,} t}{Vt}$.

Also, $$\tan D = \tan V\mu_{,} t = \frac{Vt}{\mu_{,} t}.$$

$$\therefore \quad \tan V'OX \tan D = \frac{2\mu_{,} t}{Vt} \times \frac{Vt}{\mu_{,} t} = 2. \tag{1}$$

This is the well-known equation which gives the angle made with the principal axis by the polar edge which is taken as axis of reference, when D, the inclination of the face to the base, is known.

Corollary 2. $\{110\} = -\frac{1}{2}R$; fig. 2. Each face of the rhombohedron $\{110\}$ truncates a polar edge of $\{100\} = R$. One of its faces may be taken to pass through the polar edge $V'\bar{\mu}$ of R, and may be projected in any convenient length on this edge. If the length is taken equal to $V'\bar{\mu}$, then t' is the *lower* point of trisection of the length between the apices, and O must be the lower apex of $\{110\}$. Since, to the same horizontal scale,

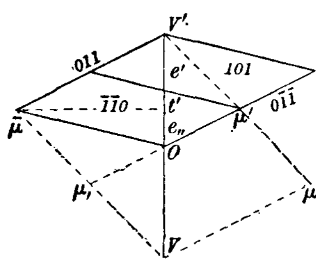

Fig. 2.

the length between the apices is one-half of VV', and the face perpendicular to the paper and projected in $O\mu'$, meets the principal axis at the lower apex, the rhombohedron is Naumann's $-\frac{1}{2}R$. The completion of the drawing needs no further explanation.

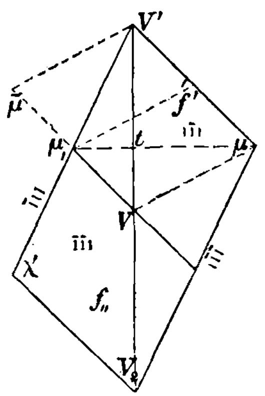

Fig. 3.

Corollary 3. $\{\bar{1}11\} = -2R$; fig. 3. This rhombohedron has each of its polar edges truncated by a face of $\{100\}$. Hence the polar diagonals $V'\mu$, $V'\mu$, may be taken for two of the polar edges of $\{\bar{1}11\}$; and the point t is then the *upper* point of trisection of the length between the apices. The lower apex is at V_2, where $VV_2 = VV'$. The completion of the figure affords no difficulty. The Naumannian symbol is clearly $-2R$; for, to the same horizontal scale, all lengths on the principal axis are doubled, and the faces are turned the opposite way to those of $\{100\}$. Its Millerian symbol is found from the fact that two of its faces are in a zone with (100); hence it is $\{1\bar{1}1\}$ or $\{\bar{1}11\}$.

Corollary 4. Similar constructions will give us further rhombohedra, the faces and edges of which either truncate, or are truncated by, those already drawn; the sign being changed with each truncation, and the length intercepted on the principal axis being successively halved or doubled. Thus we can draw $\frac{1}{4}R$, $-\frac{1}{8}R$, &c. and $4R$, $-8R$, &c.

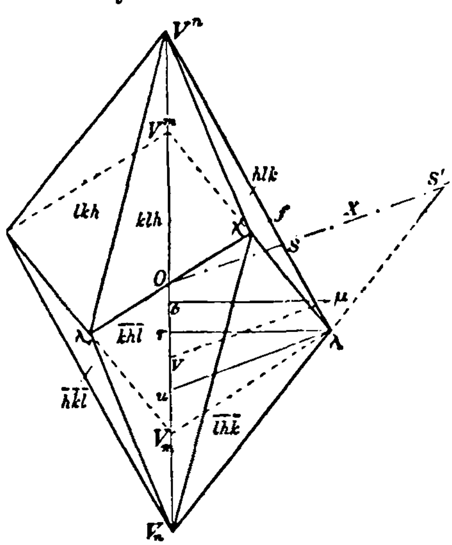

Fig. 4.

Corollary 5. $mR = \{hll\}$. If it is desired to draw this rhombohedron to the same scale as $\{100\}$, lengths mc are marked off from O on both sides of the principal axis to give the apices V^m, V_m. The length V^mV_m is trisected at τ and τ', and horizontal lines are drawn through them to meet at λ, λ', &c. the vertical lines through

μ, μ', &c. The figure is then drawn as in the cases already given. Such a figure is shown by interrupted and continuous lines $\lambda\lambda'$, &c. in fig. 4.

If m is negative, as for instance $-\frac{3}{2}R$, the points λ, λ', &c. on the horizontal lines through τ and τ' must be taken on the opposite sides of the principal axis to those in the figure. We thus have for each value of m two rhombohedra with equal angles, but in which the edges and faces are turned opposite ways. They are sometimes distinguished as *positive* and *negative*; and sometimes as *direct* and *inverse*; the first designations in each case indicating those rhombohedra in which the faces and edges are directed in the same way as those of $R=\{100\}$.

If ϕ is the angle made by the faces of mR with the base, then, from fig. 4,

$$\tan\phi = V^m\tau \div \tau\lambda = \frac{2\,mc}{3} \div \tau\lambda = \frac{2\,mc}{3} \div \mu t.$$

But, from fig 1, $\qquad \tan D = V't \div \mu t = \dfrac{2\,c}{3} \div \mu t.$

$$\therefore \qquad \tan\phi = m\tan D. \tag{2}$$

It is clear that, if m and D are known, the angle ϕ can be computed. The rhombohedron can then be drawn to any arbitrary scale by the same process as was followed in drawing the elevation in fig. 1.

We also have (see section 13, corollary 1), $m = \dfrac{h-l}{h+2l}$. $\qquad$ (3)

3. *The scalenohedron*, $mRn=\{hkl\}$; fig. 4.

When the Millerian indices are given, m and n have to be found by the relations proved in section 13:—

$$m = \frac{\theta - 3k}{\theta}, \quad n = \frac{h-l}{\theta - 3k}, \tag{4}$$

where $\theta=h+k+l$, and k is *that* index which occupies the same place in the face-symbol as zero does in the symbol of *that* face of the prism $\{10\bar{1}\}$ which truncates the median edge of the face. Now the prism-face which truncates the rhombohedral edge through O is parallel to the paper, and therefore to the axis of X. Hence zero and therefore k occupy the first place in the symbols of the prism-face and the two scalenohedral faces which intersect in this median edge. The sign of m has to be carefully attended to; for the diagonal $V^m\lambda$ lies to the right or left of the principal axis according as m is positive or negative. The sign of n is indifferent, for the length nmc has to be marked off on both sides of the origin, and the apices joined indifferently to each median coign λ.

The *auxiliary* rhombohedron mR has to be first drawn in the way described in section 2, corollary 5. The scalenohedral apices V^n, V_n, where $OV^n = nmc$, are then both joined to each median coign λ of mR.

The scalenohedron is said to be direct or inverse according as m is positive or negative. It is clear that the scalenohedron $-mRn$ must have the same angles as mRn; for it only differs in the points λ occupying positions on the opposite sides of the principal axis, but each at the same distance respectively on the lines through τ and τ' as in the case represented in the figure.

4. *The trapezohedron,* $a\{hlk\}$; fig. 5.

It is clear that figures in which the horizontal planes are projected in lines will not be satisfactory, when the base and a number of horizontal lines in, or parallel to, it have to be shown. Hence this method and Naumann's with six equal vertical spaces—both being orthogonal projections with the axis in the paper—are not suitable for hematite, quartz, and similar crystals. But the methods give a fair representation of *one* of the trapezohedra into which the scalenohedron is divided by the omission of faces connected by symmetry-planes, and a fair one of the quartz-twins with inclined axes. The present method has, moreover, the advantage of needing no computation of the lengths of the median edges.

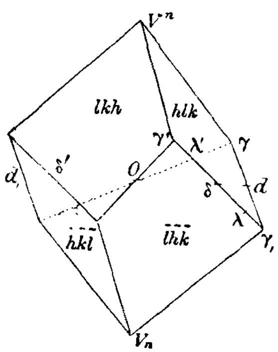

Fig. 5.

Fig. 5 represents on a diminished scale a trapezohedron derived from fig. 4 by the omission of the two central faces and the alternate faces below the paper. The side median edges have both to be extended, and the central edge $\lambda'\lambda$, has to be replaced by a new one through O in which (lkh) meets $(\bar{l}\bar{h}\bar{k})$: this edge is clearly parallel to the scalenohedral edge $V_n\lambda$ of fig. 4. The intersections of this new edge with the extended median edges give two new median coigns (γ' being one), to which polar edges are drawn. Further, the other median coigns lie on the horizontal lines through this pair, so that two of them are fixed at once. The remaining two lie on the dotted retained median edge, in which the two faces below the paper intersect. The figure is now easily completed.

The relation of the extension of the retained median edges of the scalenohedron is also easily derived from the two figures. The edge $\lambda'\lambda_{\prime}$ in fig. 4 stands parallel to the paper at distance a; and the parallel dotted line $O\gamma$ in fig. 5 is at the same distance below. It is clear therefore that the extension $\lambda'\gamma$ in fig. 5 is double the length $\lambda' f$ in fig. 4. The points γ on this median edge can therefore be easily found; and the horizontal lines through them give those on the two other retained median edges.

5. *The pyramid*, $nP2 = \{hkl\}$; fig. 6.

Where $h+l=2k$, or $\theta-3k=0$; and the same rule as to which index is k holds as in the case of the scalenohedron. It is clear that the relation of the indices makes m of equations (4) equal to zero, and there is no auxiliary rhombohedron: the median edges are horizontal.

Here the connexion between the median coigns in the equatorial plane and the apices for some easily determined pyramid has first to be found. Such a pyramid is that whose alternate polar edges are truncated by faces of the fundamental rhombohedron $\{100\}$. For the diagonals of this rhombohedron give four polar edges of the pyramid, and by joining the points B (only two of which are marked), in which these diagonals meet the central horizontal line, to the opposite apices the figure is completed.

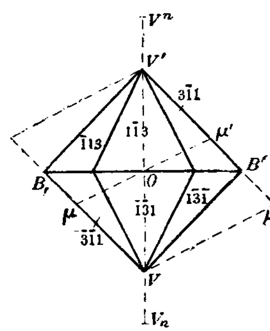

Fig. 6.

To find the Millerian symbol of the pyramid, fig. 6. Let (hkl) and (hlk) be the two faces whose polar edge $V'B'$ is truncated by (100), projected in $V'\mu$ of fig. 1. Since the three faces are tautozonal, we have by the zone-law (Lewis's 'Crystallography,' p. 40):—

$$l^2-k^2=0, \quad \therefore \; l=-k;$$

for l cannot be the same as k, since two equal indices indicate a rhombohedron, base (111), or prism $\{2\bar{1}\bar{1}\}$. Again, since $h+l=2k$;

$$h=2k-l=3k.$$

The simplest numbers which satisfy these two equations are 3, 1, $\bar{1}$; which are therefore the indices of one face. The second face is $(3\bar{1}1)$; and the pyramid is $\{31\bar{1}\}$.

Any other pyramid $nP2$ is found by joining the apices V^n, V_n, to the same points B. The length OV^n has to be found so as to accord with the scale of the pyramid $VB'B_{,}$.

According to the method of projection adopted, the pyramid-edges are drawn through different points in the equatorial plane, and the length intercepted on the principal axis varies accordingly. Thus in Lewis's 'Crystallography', p. 400, the edges pass through points M in which the polar edges of the fundamental rhombohedron meet the equatorial plane. But as shown in p. 372 of the 'Crystallography' the points B are on the lines OM and such that $OB=OM/2$. The pyramid in fig. 6 is therefore drawn to one-half the scale of that in the 'Crystallography', where its symbol is given as $2P2$. Owing to the change of scale it should now be given as $P2$, and the value of n there given, p. 402, must be halved, and to our present scale

$$n = \tfrac{1}{2}\frac{h-l}{h+l}. \tag{5}$$

Hence for $\{31\bar{1}\}$, $n=1$; and for the pyramid $\{91\bar{7}\}$ shown in the twin, fig. 15, $n=4$.

Naumann draws the pyramid-faces through the points A (see plan, fig. 1) at unit distance on the dyad axis to which the edge is perpendicular. The angle between OA and OB in fig. 1 being 30°, the line OB is in that case met at distance $OA \sec 30° = OB \sec^2 30° = \frac{4}{3}OB$. His n is therefore $\frac{4}{3}$ of that given by equation (5). The two other dyad axes $O\delta$ and OA' are met at double the distance intercepted on the first, OA; and hence the symbol $nP2$. Naumann's value of n is found in section 13, corollary 2.

6. *To draw combinations of a hexagonal prism and a rhombohedron;* figs. 7 and 8.

If the prism is $\{10\bar{1}\}$ (fig. 7), its faces truncate each a median edge of every rhombohedron, and the drawing is made in the same way for all of them. The rhombohedron, say $\{110\}$, is first drawn as in section 2, corollary 2; and the figure is then completed by drawing equal vertical lines through the median coigns: the lower ends of these lines giving points through which the lower faces pass.

When the hexagonal prism is $\{2\bar{1}\bar{1}\}$ (fig. 8), two of the faces are projected in the vertical edges marked $2\bar{1}\bar{1}$ and $\bar{2}11$. The drawing is similar whatever the rhombohedron; and this is first drawn. Let it be $\{110\}$. The faces (101) and $(1\bar{2}1)$ meet in a line parallel to the horizontal diagonal of (101), i.e. a line like AA' of the plan in fig. 1. The face $(\bar{1}\bar{1}2)$ meets (011) in the original line in which this face is projected; but it meets (101) in a new line which passes through a point in which a horizontal line through g meets the projection of (011); g being the point in which $(2\bar{1}\bar{1})$ meets the polar edge. *In this and all combinations involving a rhombohedron, the coigns at each end are of two kinds, similar*

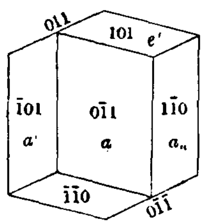

Fig. 7.

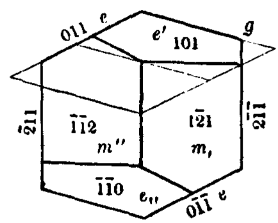

Fig. 8.

ones lying in a horizontal plane. Hence, if the position of one coign of a kind is found, the other like ones are found by drawing a horizontal line through the known coign to intersect lines on which the coigns lie. The completion of the figure is now obvious.

7. *To draw combinations of a scalenohedron, the prism* $\{2\bar{1}\bar{1}\}$, *and a rhombohedron;* figs. 9 and 10.

The method being the same whatever the scalenohedron, we select for illustration $\{20\bar{1}\} = R3$; for it gives a common crystal of calcite. This is first drawn as already explained, $\{100\}$ being the auxiliary rhombohedron. We then introduce the prism-faces, which meet at O, A, $A_{,}$, the middle points of the median edges. Through A and $A_{,}$ vertical lines are drawn, in which $(2\bar{1}\bar{1})$ and $(\bar{2}11)$ are foreshortened. These lines meet the polar edges at points which fix the two sets of horizontal lines on which like coigns lie. Each such coign is joined to the nearest pair of middle points, and fig. 9 is completed.

When the prism-edges are long, as in fig. 10, vertical lines of the required length are drawn through O, A, $A_{,}$; and the lower part is then obtained by drawing lines, parallel to those forming the lower part of fig. 9, through an apex displaced to the same extent and through new points on a horizontal line as if they were O, A, and $A_{,}$.

In fig. 10 the rhombohedron e $\{011\}$ is also represented; and the method of introducing the faces is the same for every rhombohedron. The scalenohedron and prism having been first drawn in the manner just described, through each of two points equally distant from the scaleno-

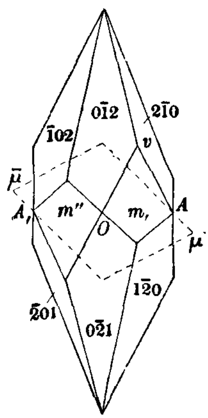

Fig. 9.

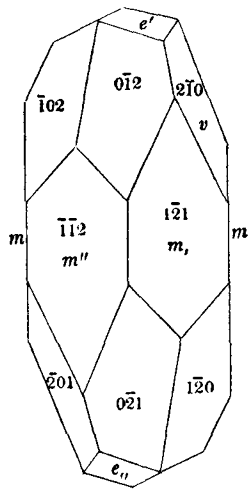

Fig. 10.

hedral apices two lines are drawn parallel respectively to the polar diagonal and edge of the rhombohedron which lie in the paper. These lines bound the complete figure; and also determine, by their intersection with the polar scalenohedral edges, the points through which horizontal lines have to be drawn so as to get the remaining coign of each of the two faces seen. The figure gives a fair representation of crystals common in Cumberland.

Calcite Twins.

8. The drawing of these twins is now easy, and the pictures are fairly satisfactory. Fig. 11 shows the common 'dog-tooth' twin of Derbyshire, having (111) for twin-face; i. e. face of association the base, and axis of rotation the principal axis. The way of drawing the figure is obvious from comparison of figs. 4 and 9.

The method does not give a satisfactory picture of a twin in which the forms are the prism $\{2\bar{1}\bar{1}\}$ with a rhombohedron; nor is it satisfactory for the similar twins of dolomite, in which the form is $\{100\}$, and the one individual is associated with the other partly along the base and partly along prism-faces.

9. In drawing twins with inclined principal axes the diagram of the simple crystal is divided by a line parallel to the diagonal in which the twin-face is projected, and perpendiculars to this line are drawn across it through each coign which is shown. Points on these perpendiculars at the same distance from the section-line as the coigns are then marked. These points are joined to one another and to the points in which the section-line is cut by edges of the simple crystal, so that the rotated or reflected portion is the symmetrical duplication of the first. The method has the advantage that, after the twin has been drawn, the figure can be turned round so as to place the twin-face vertical and the twin-axis horizontal.

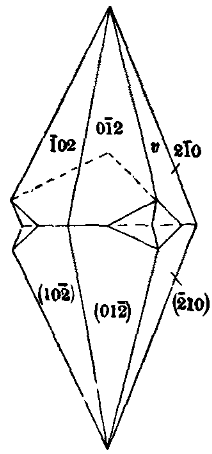

Fig. 11.—Calcite twinned on $c(111)$.

10. *Twin-law* $r(100)$. Fig. 12 shows a twin according to this law. The forms are $\{2\bar{1}\bar{1}\}$ and $\{101\}$. The twins are fairly common in Cumberland, and are often very flat from the small width of the face $m(2\bar{1}\bar{1})$ and the large extension of $(1\bar{2}1)$.

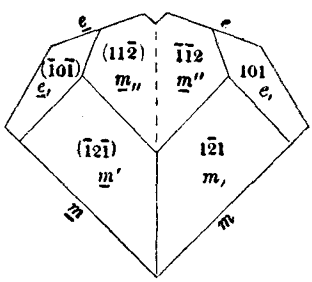

Fig. 12.

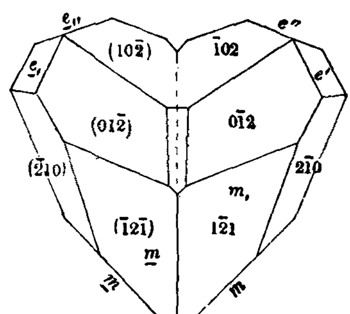

Fig. 13.

Calcite twinned on $r(100)$.

Fig. 13 shows a twin according to this law produced from fig. 10. Fig. 14 shows the common 'butterfly-twin': it has been obtained from fig. 13 by largely developing the faces $(0\bar{1}2)$ and $(02\bar{1})$ which meet in a line parallel to the polar edge $V\mu$ of fig. 1. After they had been drawn the figures have been turned round, so as to put the twin-face upright.

11. *Twin-law* e(011). This twin-law is that commonly observed in thin twin-lamellae. It is somewhat rare in distinct well-grown twins, but the Cambridge Museum has two rather remarkable specimens. The one represented in fig. 15 occurs in tabular twins of fair, and nearly equal

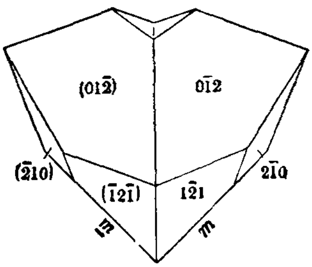

Fig. 14.
Calcite twinned on r(100).

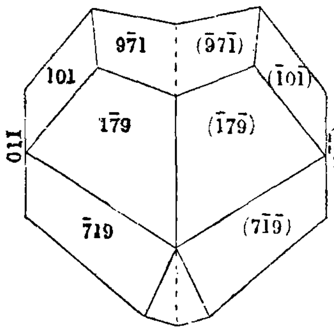

Fig. 15.
Calcite twinned on e(011).

thickness—dimensions varying from $1\frac{1}{4}$ to 2 inches across by $\frac{1}{2}$ to 1 inch thick. Their surfaces have a thin crust of pitted and frosted calcite, which gives no reflected image: they are associated with some decomposing marcasite, and one has the cast apparently of a cubic face of fluor crystal. At first sight the forms seem to be those shown in fig. 7; but close inspection of the apparent prism shows that the edges are not parallel, although nearly so. As one of the twins was slightly broken, it was possible to get from a fragment the approximate angles of one of the faces to the three cleavages by gumming a piece of cover-glass to the rough face. The observed angles, and those computed for $\{91\bar{7}\}$ by formulae (25) in this Magazine, vol. xii, p. 339, are given in the following table:—

		Computed.		Measured.
$\lambda = 100 \wedge 91\bar{7}$	...	41° 42′	...	—
$\bar{1}00 \wedge$ „	...	138 18	...	138° 7′
$\mu = 010 \wedge$ „	...	81 13	...	81 21
$0\bar{1}0 \wedge$ „	...	98 47	...	98 33
$\nu = 001 \wedge$ „	...	116 12	...	116 20
$00\bar{1} \wedge$ „	...	63 48	...	63 41

For the determination of the face-symbol from measurements of the angles λ, μ, ν, equations (25) can be easily transformed to give the following:—

$$\frac{h-k}{k-l}=\frac{\cos\lambda-\cos\mu}{\cos\mu-\cos\nu}=\frac{\sin\frac{\lambda+\mu}{2}\sin\frac{\lambda-\mu}{2}}{\sin\frac{\mu+\nu}{2}\sin\frac{\mu-\nu}{2}}. \tag{6}$$

Introducing the observed values of the angles, we find by tables of natural, or logarithmic, sines and cosines:—

$$\frac{h-k}{k-l}=1, \text{ very nearly };$$

$$\therefore\quad h-2k+l=\theta-3k=0;$$

and the form is a bipyramid. The indices are then easily found from any pair of equations (5).

The angles λ, μ, ν, made by any face with the axial planes of a rhombohedral crystal are connected with the angle $D=100 \wedge 111$ and the angle, CP, between that face and the basal plane, by the following expression (which may be readily deduced in a manner similar to that in Lewis's 'Crystallography', p. 355):—

$$\cos\lambda+\cos\mu+\cos\nu=3\cos D\cos CP. \tag{7}$$

From equations (25) we can now obtain the following equations, which render easy the determination of h, k, l:

$$\frac{h}{\cos\lambda\cos D-\cos CP\cos a}=\frac{k}{\cos\mu\cos D-\cos CP\cos a}$$
$$=\frac{l}{\cos\nu\cos D-\cos CP\cos a}=\frac{\theta}{(1-\cos a)\cos CP}; \tag{8}$$

a being the angle over the polar edge of $\{100\}$.

The second twin, of which there are three or four individuals, occurs on a specimen from the lead mines of Seven Churches, Co. Wicklow. They are very thin; and the largest are about $1\frac{1}{4}$ inches across by only $\frac{1}{6}$th of an inch in the thickest part. Most of them are now detached, as they were very loosely attached by the edge to the matrix. The form is the scalenohedron $\{13.\ 0.\ \overline{11}\}=R12$, a form which Irby[1] gives as doubtful. Owing to the edges being partially broken along the cleavages, it was possible to measure the several zones; and the angle $(100):(13.\ 0.\ \overline{11})$ has a mean value of 46° 21′ (the extremes in different zones being 46° 18′ and 46° 23′). The computed angle is 46° 23′. Several of the faces, and especially the widely extended ones, gave more than

[1] J. R. McD. Irby, 'On the crystallography of calcite.' Inaug.-Diss., Bonn, 1878.

one image. These twins are associated with very long, slender crystals, which proved on measurement to be the simple scalenohedron $\{13.0.\overline{11}\}$.

As is common in twins according to this law which are known to me, there is in parts an overlapping of one individual over the other in both the twins described.

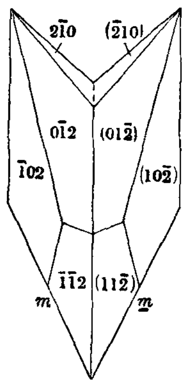

Fig. 16.—Calcite twinned on $f(\bar{1}11)$.

12. *Twin-law* $f\{\bar{1}11\}$. Twins according to this law are rather uncommon. They sometimes occur in Cumberland in crystals in which the prism $\{2\bar{1}\bar{1}\}$ is largely developed; and they have been found in Somersetshire[1] in rough crystals which are a combination of $\{20\bar{1}\}$ with $\{2\bar{1}\bar{1}\}$. A twin of this latter kind is shown in fig. 16. To draw it we have only to divide a crystal, like that shown in figs. 9 or 10, by a line parallel to $V_2\mu$ of fig. 3, and to erect perpendiculars through the coigns to this line.

13. The principle of the drawings lends itself to the determination of certain numerical relations; but not to the computation of angles or the converse. We shall illustrate the way in which the drawings may be useful, by deducing the relations between Naumann's and Miller's symbols for the scalenohedron and other forms.

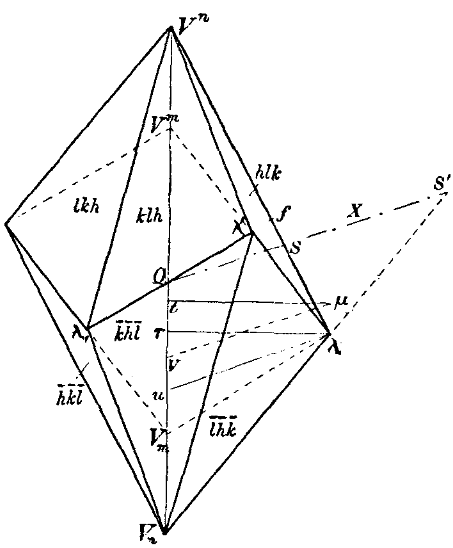

Fig. 4 (repeated).

To determine in the scalenohedron, $mRn=\{hkl\}$, the relations between m and n and the indices, h, k, l of one of its faces.

It is easy to see that the Millerian symbols inscribed on the scalenohedral faces are consistent. For two faces meet in the polar edge $V^n\lambda$ and intersect OX at S (fig. 4), where $OS=V\mu/h$. If the face above the paper is denoted by (hlk), that which is symmetrical to it and behind the paper is (hkl), and the parallel face (that to the left and bottom) is $(\bar{h}\bar{k}\bar{l})$.

[1] H. L. Bowman, Min. Mag., 1903, vol. xiii, p. 329.

Draw $u\lambda$ parallel to $V\mu$ and OX to meet the principal axis at u, then $u\lambda = V\mu$; and $Vu = \mu\lambda = t\tau = \frac{m-1}{3}c$.

Also,
$$Ou = OV + Vu = \frac{m+2}{3}c.$$

From the similar triangles OSV^n, $u\lambda V^n$, we have

$$h = \frac{V\mu}{OS} = \frac{u\lambda}{OS} = \frac{uV^n}{OV^n} = \frac{OV^n + Ou}{OV^n}$$

$$= \frac{mnc + \frac{m+2}{3}c}{mnc} = \frac{m+2+3mn}{3mn}. \tag{9}$$

Again, the two faces which meet in $V_n\lambda$ have $-l$ for first index; i. e. $OS' = -\frac{V\mu}{l}$. Hence from the similar triangles $OS'V_n$, $u\lambda V_n$, we have

$$-l = \frac{V\mu}{OS'} = \frac{u\lambda}{OS'} = \frac{uV_n}{OV_n} = \frac{OV_n - Ou}{OV_n}$$

$$= \frac{mnc - \frac{m+2}{3}c}{mnc} = \frac{3mn - m - 2}{3mn};$$

$$\therefore \qquad l = \frac{m+2-3mn}{3mn}. \tag{10}$$

Again, the two faces (klh) and (khl) will, if produced, meet in a line in the paper parallel to $O\lambda'$; and it will therefore meet OX on the *negative* side in a point E (not shown). Now $O\lambda'$ is parallel to $V_m\lambda$; consequently the triangle OV^nE is similar to $u\lambda V_m$.

$$\therefore \quad -k = \frac{V\mu}{OE} = \frac{u\lambda}{OE} = \frac{uV_m}{OV^n} = \frac{OV_m - Ou}{OV^n}$$

$$= \frac{mc - \frac{m+2}{3}c}{mnc} = \frac{2(m-1)}{3mn};$$

$$\therefore \qquad k = \frac{2(1-m)}{3mn}. \tag{11}$$

From equations (9), (1), and (11) we have

$$\frac{h}{m+2+3mn} = \frac{k}{2(1-m)} = \frac{l}{m+2-3mn} = \frac{1}{3mn}. \tag{12}$$

Hence we have

$$\frac{h+l-2k}{h+k+l}=\frac{2(m+2)-4(1-m)}{6}=m. \qquad (4^*)$$

and
$$\frac{h-l}{h+l-2k}=\frac{6mn}{2(m+2)-4(1-m)}=n. \qquad (4^{**})$$

Also
$$mn=\frac{h-l}{h+k+l}. \qquad (13)$$

As has already been said, the sign of m has to be carefully attended to; that of n is immaterial.

Corollary 1. When $n=1$, the scalenohedron coincides with the auxiliary rhombohedron mR. From equation (4^{**}), it follows that $k=l$. Hence from equation (4^*),

$$m=\frac{h-l}{h+2l}. \qquad (3^*)$$

Corollary 2. For the pyramid n_1P2, the median edges of the scalenohedron, which pass through points A on the dyad axes at distance a from the origin, become horizontal, and the height mc of the auxiliary rhombohedron is zero. Hence $m=0$; i. e. $h+l-2k=0$. (14)

But the distance OV'' is finite. In the pyramid it is denoted by n_1c; and in the scalenohedron mnc: $\therefore n_1=mn$.

Therefore from equation (13)

$$n_1=mn=\frac{h-l}{h+k+l}.$$

Introducing the value of k from (14),

$$n_1=\frac{2}{3}\frac{h-l}{h+l}. \qquad (15)$$

This is the relation corresponding to Naumann's way of drawing the pyramid; for, as stated in section 5, he draws each horizontal edge through a point A on the perpendicular dyad axis to meet each of the two adjacent dyad axes at a distance $2a$. The scale in drawing fig. 6 is $\frac{3}{4}$ of his, and consequently n of section 4 is given by

$$n=\tfrac{3}{4}n_1=\tfrac{1}{2}\frac{h-l}{h+l}. \qquad (5^*)$$

The relation (5) can easily be deduced from fig. 6 by drawing the axis OX and a parallel line through B, and employing similar triangles in a manner corresponding exactly with that by which the scalenohedral relations have been obtained.

14. As already pointed out in section 4, $\lambda'\gamma$ (on the dotted line $O\gamma$ in fig. 5) is equal to $2\lambda'f$ of fig. 4, when both figures are drawn to the same scale. We can now find the extension in terms of the indices. Now $O\lambda'$ and $V_m\lambda$ being parallel edges of mR, $O\lambda' = \frac{1}{2}V_m\lambda$.

Then by the similar triangles V^nOf, $V^nV_m\lambda$, fig. 4, we have,

$$\frac{Of}{V_m\lambda} = \frac{OV^n}{V^nV_m} = \frac{OV^n}{OV^n + OV_m} = \frac{mnc}{mnc + mc} = \frac{n}{n+1}. \tag{16}$$

$$\therefore \quad \frac{\lambda'f}{V_m\lambda} = \frac{Of - O\lambda'}{V_m\lambda} = \frac{Of}{V_m\lambda} - \frac{O\lambda'}{V_m\lambda} = \frac{n}{n+1} - \frac{1}{2} = \frac{n-1}{2(n+1)}.$$

$$\therefore \quad \lambda'\gamma = 2\lambda'f = \frac{n-1}{n+1}V_m\lambda = \frac{1}{2}\frac{n-1}{n+1}O\lambda'. \tag{17}$$

From equation (4) we also have

$$\frac{n-1}{n+1} = \frac{h-l-\theta+3k}{h-l+\theta-3k} = \frac{k-l}{h-k}.$$

$$\therefore \quad \lambda'\gamma = \frac{k-l}{2(h-k)}O\lambda'. \tag{18}$$

Care must be taken in using (18) to attend to which of the three indices is k.