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On a protractor for use in constructing stereographic and gnomonic projections of the sphere.

(*With Plates III, IV, IV a, and V.*)

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THE protractor of which an account is given in the following pages is intended to facilitate the construction of stereographic and gnomonic projections of the sphere, especially on the part of students of crystallography. The fundamental properties of these projections, on which its design depends, have long been known, and have been the basis in the past of methods somewhat similar to those about to be described. Any novelty there may be in the device is to be sought therefore in the convenient arrangement of its parts, and in the ease with which it may be constructed and used, rather than in the principles which it embodies. We will consider first the construction of the protractor, then its application in the case of the stereographic and gnomonic projections, and in conclusion will deal with a few points of interest in the history of these projections.

Construction of the Protractor.

The protractor (plate III, fig. 1) consists of a strip of boxwood or other suitable materal, of any convenient length and thickness, and of a breadth exactly equal to the radius of the sphere to be projected.

When selecting the radius for this sphere, the length of 10 centimetres naturally suggests itself, and this is doubtless the best unit to choose when it is desired to secure all the accuracy of which a graphical method is capable. It has too the advantage, that stereographic nets based on this unit have been widely employed and are readily obtainable. For students, however, so large a radius is undesirable, and after careful consideration it has been found best to select as unit a radius of 2·5 inches. In constructing a projection on this scale a sheet of paper of only moderate size is required and the necessary circles can in most cases be described with an ordinary pair of compasses; for those of very long radii recourse must be had to beam-compasses, or better, to the curved ruler designed by G. Wulff and E. S. Fedorov. A primitive circle of 5 centimetres radius may be employed if preferred but has been found by experience to be somewhat too small. On the other hand, a radius greater than 10 centimetres will be found inconveniently large. Between these limits any value may be adopted, as for instance, the 7 centimetre radius chosen by the late S. L. Penfield, but, as already stated above, 2·5 inches (6·35 centimetres) has proved to be the most convenient radius for general use.

A protractor adapted to this radius must be 2·5 inches wide and should be about 12 inches long.[1] A zero-line OZ (fig. 1) is drawn across it at right angles to its length, at a distance from one end rather greater than the width of the protractor. Taking as centre the point Z, where this zero-line intersects one edge, the opposite edge ST is graduated by means of a circular dividing engine. The portion of the scale extending from O to T is divided to degrees and the divisions numbered as shown in plate III, fig. 1. The divisions from O to S are exactly the same as

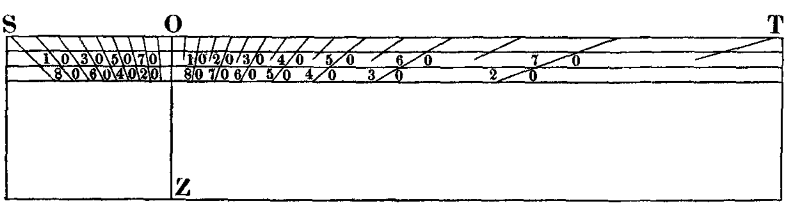


Fig. 1.

those from 0° to 45° along OT, but are numbered differently, each division between O and S representing 2° instead of 1°. A protractor for a

[1] Protractors graduated on boxwood for radii of 2·5 inches and 10 centimetres can be obtained from Mr. W. H. Harling, 47 Finsbury Pavement, London, at a cost of 5*s.* and 8*s.* 6*d.* respectively.

radius of 10 centimetres must be 10 centimetres wide and in this case OT may be divided to half degrees and OS to degrees. For the sake of clearness the finer divisions have been omitted in plate III, fig. 1, as also in text-fig. 1.

From the method of construction it appears at once that the distance from O of any division lying between O and T is numerically equal to the tangent of the corresponding angle, read on the upper row of numbers, when measured with a scale for which the radius of the primitive is taken as unit. Such a scale is provided on the back of the protractor. If the radius of the primitive is 2·5 inches, the scale is divided to fortieths of an inch, while the protractor for use with a primitive of 10 centimetres radius has a millimetre scale. Similarly the distance from O of any division on the scale OS is numerically equal to the tangent of half the angle as read on the lower row of numbers, and the divisions themselves correspond to degrees stereographically projected on a diameter. The use of the protractor will be facilitated if both edges are graduated as shown in plate III, fig. 1, though this is unessential.

Application to the construction of stereograms.

Since the graduation of OT is identical with that of the ordinary rectangular protractor, this scale may be employed in the usual way for setting off angles.

So far as special applications are concerned, the two scales OS and OT taken separately, or in conjunction, enable us to solve with ease the following problems of construction:—

(1) To project on any diameter the position of a point of which the angular distance from the primitive is known.

(2) To draw through any projected point a great circle with diameter at right angles to the line joining the given point to the centre.

(3) To find the pole of any projected great circle; or, in general, to find the point 90° removed from a given point and lying in the same diameter.

(4) To find the projected position of a face parallel to a face whose projected position is known. In other words, to find the point diametrically opposite to any given point.

(5) To draw small circles about any point.

These problems will now be considered in detail.

(1) *To project on any diameter the position of a point P (fig. 2) of which the angular distance, θ, from the centre, or $90°-\theta$ from the primitive, is known.*

Describe the primitive circle *ZSB*. Place the protractor along the given diameter with its zero-point *O* on the centre of the primitive. The position of *P* can then be read off directly on the scale *OS* as shown in fig. 2, where $\theta = 40°$.

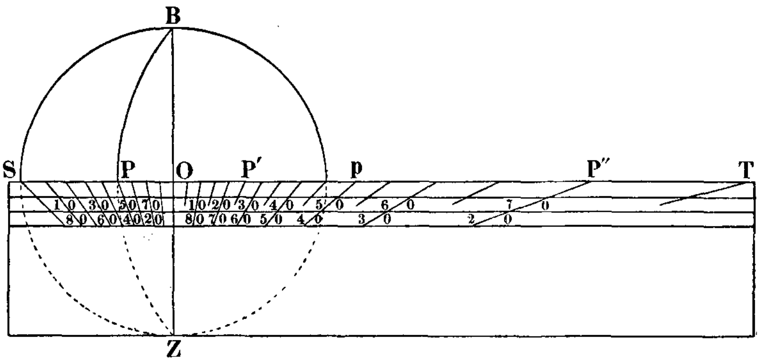


Fig. 2.

(2) *To draw through any point P a great circle BPZ perpendicular to that projected in OP.*

With the protractor in the same position as above (fig. 2) read the position of *P* on the scale *OS* using either row of numbers, and then find a point *p*, which has the same reading (50° upper row, 40° lower row) on *OT*, that *P* has on *OS*. The circle described with centre *p* and radius *pP* passes through *B* and *Z* and is the great circle required.

This construction depends on a well-known property of the stereographic projection, namely, that the radius *pP* of any great circle *BPZ* is numerically equal to r cosec θ, where r is the radius of the primitive and θ the north polar distance of *P*, in other words where $90° - \theta$ is the elevation of the plane of the great circle above the plane of the primitive. Its truth may be demonstrated geometrically in the following simple manner :—

In fig. 3 let *P* be the projection of a point of which the north polar distance is measured by the angle θ. Draw *ZOB* perpendicular to *OP*. Join *ZP*. Bisect *ZP* in *D* and draw *Dp* perpendicular to *ZP* cutting *PO* produced in *p*. Join *Zp*. Now *p* is the centre of a circle passing through the three points *Z*, *P*, and *B*, and represents in the projection the great circle containing *P* and the diameter *ZB*. But the angles marked ϕ, viz. *PZO*, *PpD*, and *DpZ* are all equal. Further $2\phi = \theta$, for the eye placed at *Z* sees *N* projected at *P*, and the angle $NOB = \theta$. We have, therefore, $pP = pZ = OZ \operatorname{cosec} 2\phi = r \operatorname{cosec} \theta$.

Now $\operatorname{cosec} \theta = \cot \theta + \tan \frac{\theta}{2}$; and it will be seen from fig. 2 that

$$Op = r \tan(90° - \theta) = r \cot \theta, \text{ and that } OP = r \tan \frac{\theta}{2},$$

hence $pP = r \operatorname{cosec} \theta$, and therefore the circle described with centre p and radius pP is the great circle required.

(3) *To find the pole P' of the great circle BPZ.*

Keep the protractor in the same position as before (fig. 2). Take the reading of P on the scale OS, using the *upper row of numbers*, divide this reading by two, and so find on OT, *upper row of numbers*, the pole P' of the great circle BPZ. In fig. 2 the reading of P is 50°, the corresponding position of P' is therefore at 25° on OT.

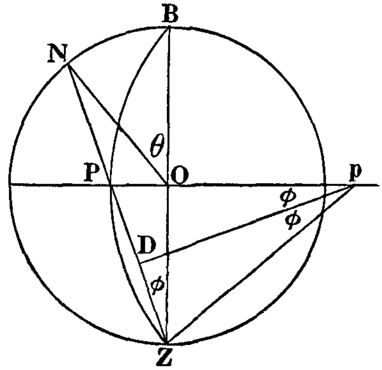


Fig. 3.

The rationale of this process follows at once from a consideration of fig. 2. The point P is at 40° from the centre, measured stereographically, while P' is at 25° on OT. But 25° on OT correspond to 50° measured on OS and $40° + 50° = 90°$.

(4) *To find the point P'', the projection of a point diametrically opposite to that projected in P.*

The protractor still remaining in the same position (fig. 2), read the place of P on the scale OS, using this time the *lower row of numbers*. Divide the reading obtained by two and so locate P'' by finding the corresponding reading on the *lower row of numbers* of the scale OT.

In this case again the reason for the procedure employed can readily be deduced from an inspection of fig. 2. The point P is at 40° from the centre measured stereographically, and P'' is at 70° from O on OT. But $70° \times 2 + 40° = 180°$.

(5) *To draw a small circle of given angular radius $\phi°$ round any point P.*

In crystallographic work two cases present themselves: (i) the point about which the small circle is to be described lies on the primitive, the small circle is accordingly in this case a small vertical circle ; (ii) the point is situated within the primitive.

(i) To describe round P (fig. 4) the small circle QOR with angular radius $\phi°$:—

Place the protractor along the diameter through P so that the reading for ϕ taken on the scale OS, *upper row of numbers*, falls on the centre C of the primitive. (In fig. 4, $\phi = 50°$.) Mark the position of O, the zero-point of the protractor, and note on OT, *upper row of numbers*, the position of the point p, which has the same reading, $\phi°$, on OT that C

has on OS. The circle QOR described with centre p and radius pO is the small circle required.

From a consideration of fig. 4 it will readily be seen that O is the projected position of a point of which the north polar distance is $90°-\phi°$ and which lies, therefore, at $\phi°$ from the point P on the primitive. Now it can be shown that the radius of a small circle described at an angular distance $\phi°$ from a point P in

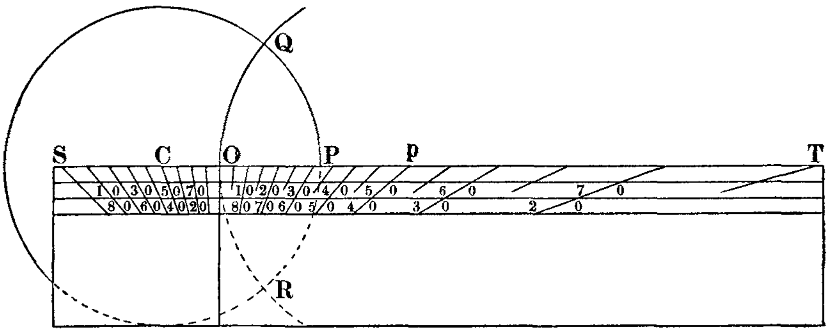


Fig. 4.

the primitive is equal to $r \tan \phi$, where r is the radius of the primitive, for its diameter is equal to $r \tan(45° + \frac{\phi}{2}) - r \tan(45° - \frac{\phi}{2}) = 2r \tan \phi$. But from the construction of the protractor $r = 1$, and $Op = \tan \phi$. Hence, therefore, the circle ROQ described about p with the radius pO is the small circle required.

If the point P lies inside the primitive the method of procedure will vary with its position.

(ii a) The point P lies at the centre of the primitive.

In this case the angular radius ϕ of the small circle is marked off directly from the scale OS and its centre coincides with P.

(ii b) The given point does not lie at the centre of the primitive but occupies some other position, such as P, fig. 5.

In this case the protractor is placed on the diameter through P with its zero-point on the centre of the primitive and the distance of P from the centre is measured with the scale OS. Two points Q and R are now found both at the given distance ϕ from P. The circle through Q and R described with centre p, a point half-way between them, is the small circle required. In fig. 5, P is at 50° from the centre of the primitive and the value of ϕ is 30°. The point Q lies at $50° + 30° = 80°$ from O, while R is at $50° - 30° = 20°$ from O. QR is bisected at p. For making this bisection the scale on the back of the protractor will be found of service.

If owing to the position of P, or to the value of the angle ϕ, the points Q and R lie outside the limits of the semidiameter, it will be necessary to employ the scale OT as well as, or instead of, the scale OS, remembering that each degree read on OT corresponds to two degrees read on OS. This case is also illustrated in fig. 5. Here P' is situated

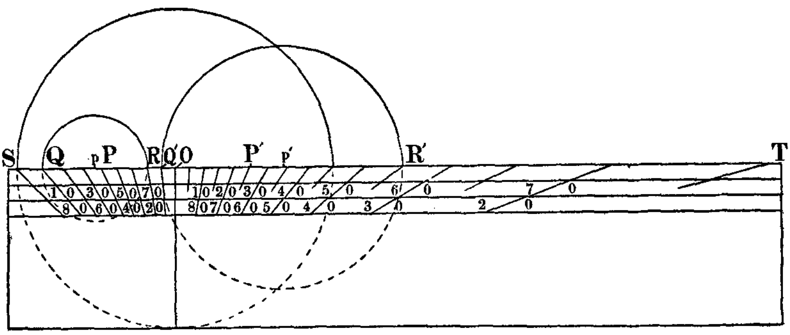


Fig. 5.

at 50° from the centre and $\phi = 60°$. It will be observed that P', the projection of a point 50° from the north pole, reads 25° on the scale OT. The point Q' at 60° from P' is found by marking off 10° on the scale OS, for $50° + 10° = 60°$. The point R', also at 60° from P', is found at 55° on OT, for $\frac{1}{2}(50° + 60°) = 55°$. The centre of the small circle is then obtained by bisecting $Q'R'$ in p'.

The measurement of angles on a stereogram.

When a stereogram has been made it will often be found desirable (i) to measure the angles between two crystal-faces of which the projected positions have been plotted, or (ii) to determine the angles between zones which intersect at a certain point.

The second of these two problems may readily be solved by drawing tangents to the zone-circles at the point where they meet and measuring the angle between these tangents. The solution depends on a valuable property of the stereographic projection, namely its angular truth; in other words the angles between great circles on the sphere are preserved in the projection. A convenient construction is given by S. L. Penfield.[1]

The measurement of the angle between two points necessitates the

[1] S. L. Penfield, Amer. Journ. Sci., 1901, ser. 4, vol. xi, p. 19.

construction of the great circle passing through the two points and the determination of the pole of this great circle. The protractor will be found useful in carrying out both these operations. When the great circle and its pole have been determined, the angle between the two points can be found in the usual way by measuring the arc of the primitive cut off by two lines drawn from the pole through the given points. Conversely this construction may be employed to plot the positions of faces lying in a given zone at known distances from some point in the zone.

This problem may be also more rapidly and conveniently solved by means of a stereographic net, as has been pointed out by several writers. Plate IV is a semicircular net for a primitive of 2·5 inches radius, the great and small circles being drawn 2° apart. If a stereogram is drawn on tracing-paper it may be placed above the net with the two centres coincident and the angular distance between any two points may be determined by rotating the drawing above the net about a needle passing through the common centres until the two points lie along the same great circle or evenly between two adjacent ones. The angular distance between the points may then be read off at once by means of the small circles. If, however, the stereogram has been drawn on ordinary paper the problem can be conveniently solved by means of a triangular compass, the three legs of which are placed on the centre of the primitive and on the two points respectively. When set, the compass is transferred to the net, the same leg being placed on the centre in each case, and rotated about this leg until the extremities of the legs which rested on the two given points lie on the same great circle. The angle between them is then read off as before. If such compasses are not at hand, a piece of tracing-paper will answer very nearly as well. It is placed first on the drawing, and the positions of the centre and of the two points are marked on it. It is then transferred to the net. Most convenient of all is a net printed on good transparent tracing-paper, plate IV a, which may be placed directly on the drawing. This will be found very useful, not merely in making measurements on the completed stereogram, but also during the actual construction, as points can be pricked through direct from the net, which may be discarded for a new one as soon as it shows signs of wear.

Application to the gnomonic projection.

Points marked off by means of the protractor using the scale *OT*, upper row of numbers, lie at distances from the zero-point numerically equal to the tangents of the corresponding angles when the radius of the sphere of reference is taken as unity. It follows therefore that the scale OT may be used for projecting gnomonically points of which the north polar distances are given, when the north polar point itself is placed at the centre of the projection. The protractor will also be found of service in determining the angle between any two projected faces as well as the angles between zones.

To find the angle between two faces of which the projections are given.—Let A and B (fig. 6) be the two given points and O the centre of the projection. Join AB. Then AB is the projection of a great circle and represents the zone containing A and B. Draw ON perpendicular to AB or to AB produced. Place the protractor with its zero-point on O and with the scale OT coincident with ON. Read off the position of N on OT, using the upper row of numbers. In NO produced find the point N' which has the same reading on OS, upper row of numbers, that N has on OT.[1] Then $AN'B$ is the angle required. The converse of this construction may be employed to mark off a succession of points lying in the same zone at given angular distances from some point in the zone of which the position is known.

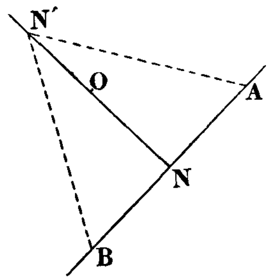


Fig. 6.

The validity of the above construction may be proved as follows:—Let O (fig. 7) be the centre of the projection, C the centre of the sphere, A and B the projections of two faces inclined to one another at the angle BCA. Draw ON perpendicular to BA and join NC. Produce NO to N', making $NN' = NC$. The angles BCA, $BN'A$ are equal, and the planes CNN', $AN'B$ are at right angles, while OC is equal to r, the radius of the sphere. Now $CN = r \sec \theta$, where θ is the angle NCO, and

$$\sec \theta = \tan \theta + \tan \tfrac{1}{2}(90° - \theta).$$

But by construction $NN' = CN$. Hence

$$NN' = r \tan \theta + r \tan \tfrac{1}{2}(90° - \theta).$$

Now in using the protractor in the manner described above (fig. 6), the distance $ON = \tan \theta$ and $ON' = \tan \frac{1}{2}(90° - \theta)$, for, from the construction of the protractor, distances measured from O along OT are equal to the tangents of the angles read off on the upper row of numbers of that scale, and distances measured from O along OS are equal to the tangents of angles exactly half those read on the lower scale

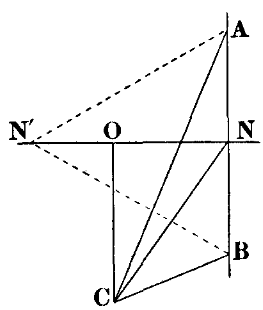


Fig. 7.

[1] The point N' is the 'Winkelpunkt' of V. Goldschmidt, and may be termed the 'angle-point' of N.

of OS, and therefore equal to half the supplements of the angles read on the upper row of numbers. Thus if N reads 60° on OT, N' must be taken at 60° on OS, upper scale. ON' is therefore $= \tan \frac{1}{2}(90° - 60°) = \tan 15°$.

To find the angle between two zones.—Let O be the centre of the projection (fig. 8) and let the projected zones be OF and GF intersecting at F. It is required to find the angle between the zones.

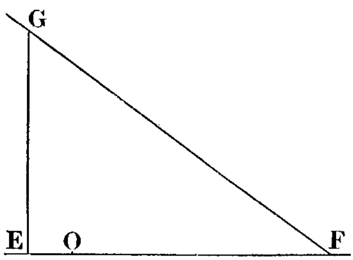


Fig. 8.

Place the protractor along OF with its zero-point on O and read off on the scale OT, *upper row of numbers*, the position of the point F.

In FO produced find the point E which has the same reading on OS, *upper row of numbers*, that F has on OT. From E draw EG perpendicular to EF cutting FG in G. Place the protractor along EG with its zero-point at E, then the position of G, as read off on the scale OT, gives the angle between the zone-lines OF and GF.

Proof.—The validity of this very simple construction may be shown as follows:—In fig. 9, let O be the centre of the projection and C the centre of the sphere of reference, and let the two zone-lines OF and GF intersect at F. Join FC, and from C draw CH perpendicular to CF. Bisect the angles OCH and OCF by CE and CW, meeting HF in E and W respectively. From H and E draw perpendiculars to HF, meeting FG in D and G respectively. In EF take ER equal to OC, the radius of the sphere. Join DW and GR. The planes DFH and HFC are at right angles, and since CH is perpendicular to CF, F is the pole of the zone projected in HD, and the angle between the zones GF and OF is measured by the angle between the points H and D.

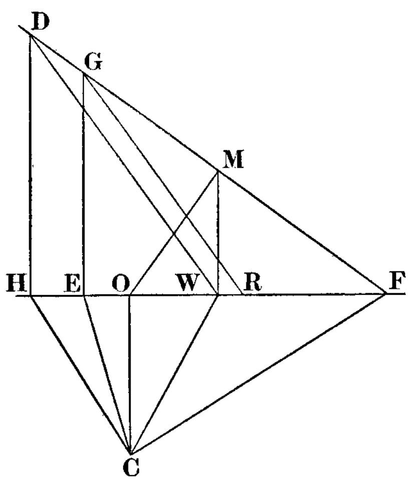


Fig. 9.

Now from the preceding proposition DWH is the angle between D and H, for by construction the angle OCW is half the complement of the angle OCH, and therefore W is the 'angle point' of H. Similarly, E is the 'angle point' of F. Further, $HC = HW$ and $FE = FC$, for the angles HCW and CWH are equal, as also the angles FEC and ECF. But $DH : GE = HF : EF = HF : CF = HC : CO = HW : ER$.

Hence $DH : HW = GE : ER$, and therefore the angles GRE and DWH are equal.

Now when the protractor is used as described above, the graduations on the scale OT in contact with EG are set off from the centre R. The position of E on the scale OT gives, therefore, at once the angle between the zones.[1]

The following is an alternative construction which may be found useful in some cases, though it is a little more troublesome to carry out.

As before, let O be the centre of the projection (fig. 10) and OF and GF the projected zone-lines. Place the protractor along OF with its zero-point on O and read off on the scale OT, *upper row of numbers*, the position of the point F. Divide this reading by two and so obtain the position of W ('stereographischer Punkt' of V. Goldschmidt). Draw WM perpendicular to OF and join OM. Then MOW is the angle required.

Proof.—In fig. 9, the point W represents the point of the same name in fig. 10. Draw WM perpendicular to OF and join OM. We have to show that the angles DWH and MOW are equal.

Now the angle OCF is bisected by CW, COF is a right angle, and $HW = HC$, therefore

$$WF : OW = CF : CO = CH : OH$$
$$= HW : OH = HW + WF : OH + OW$$
$$= HF : HW, \text{ and consequently}$$

$HW : OW = HF : WF$. But $HD : MW = HF : WF$, therefore $HD : HW = MW : OW$, and hence the angles DWH and MOW are equal.

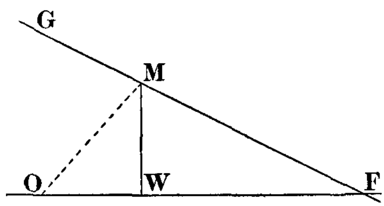


Fig. 10.

In the case considered above one of the zone-lines passed through the centre of the projection, but the construction is equally applicable to the general case in which this does not occur, for it is only necessary to join the point of intersection of the two zones to the centre and proceed as described.

To facilitate the measurement of angles on the completed projections the protractor may be constructed of celluloid. Such a protractor is shown of half-size in plate III, fig. 2. It bears the same graduations as the boxwood protractors but is double their width, and in the case of the scale OT the graduations for every fifth degree are continued to as near the centre as possible, and between 80° and 85° the graduations for every degree are so continued. A scale of equal parts, 1 div.$=0{\cdot}01$ radius, runs down the centre of the protractor. For the sake of clearness the finer divisions have been omitted in the

[1] This construction may also be utilized to find the angle between two given points. For let H and D be the two points. Find F the pole of the zone HD and join DF. Then find E and proceed as above. Another construction, due to E. S. Fedorov, is given on page 105.

plate. Being transparent the protractor can be used for reading off angles, as well as for performing all the other operations described above.

Another device which will also be found useful is a modified form of triangular compass by the aid of which the problem of finding the angle between any two projected poles in the gnomonic projection may readily be solved graphically with a fair degree of accuracy. This instrument (plate V) consists of two metal bars of rectangular section pivoted together by means of a good sector joint thus enabling the bars to be set at any angle. Metal sliding pieces carrying stout steel pins move freely on the bars and can be clamped in any position. One bar carries two short pins of equal length, the other bar carries one long pin. The pins are so arranged that when the instrument rests on the three points A, B, and O, the three pins are all perpendicular to the plane ABO, and the plane drawn parallel to ABO through the points a and b cuts off a length co from the longer pin, exactly equal to the radius of the sphere of reference. Pins of various lengths can be provided. In practice it is found convenient to have two long ones, in which the distances co are 2·5 inches and 10 centimetres respectively, for use with projections constructed by means of the corresponding protractors.

To find the angle between any two faces all that is necessary is to arrange the instrument so that the point O rests on the centre of the projection while the points A and B fall on the two projected poles. When this is done it is clear that the angle bca is the angle between the two faces, since the points a and b are the projections on the plane oab of two faces inclined at an angle bca and A, B, and O are vertically below a, b, and o. The angle bca can readily be measured, for on inverting the instrument so that the points a, b, c rest on a sheet of paper their positions can be marked and the angle bca read off by a protractor. In this way it is possible to check graphically the results of calculation without drawing any new lines or in any way interfering with a finished diagram.

After the protractor described in the first section of this paper had been designed, and its value tested by actual use, a paper by E. S. Fedorov, entitled 'Die Wichtigkeit der Anwendung des stereographischen Lineals', came to my notice through the medium of an abstract.[1] In this paper, which appeared originally in Russian,[2] an account is given of an instrument in which the same principles are utilized as those employed by Penfield and myself, although it

[1] Zeits. Kryst. Min., 1908, vol. xliv, p. 89. See also ibid., 1903, vol. xxxvii, p. 141. [2] Annuaire Géol. Min. Russie, 1905-1906, vol. viii, p. 26.

differs considerably from the protractor described above in the way these principles are applied, and especially in the arrangement of the scales. One of these rulers, recently obtained from the firm of Fuess, consists of a strip of German-silver, 3 centimetres wide and 41 centimetres long. On one edge is engraved a scale of stereographically projected degrees extending to 90° on the one side of zero, and to 140° on the other; this scale is, in fact, identical with Penfield's scale No. 3, except that it is intended for use with a primitive of 10 centimetres radius, whereas Penfield employed a radius of 7 centimetres. The other edge is occupied partly by a scale of millimetres, 0 to 200, and partly by a scale giving the length of the radii of small circles measured from the primitive. The graduation is beautifully executed, and the instrument is capable of giving very accurate results; it is, however, somewhat costly.

In the same paper several problems of the gnomonic projection are also discussed, and a construction is given for finding the angle between two poles in a zone. As this construction can easily be carried out with the aid of the stereographic protractor, and on account of its elegance and accuracy, deserves to be widely known, it seems desirable to reproduce it here:—

Fig. 11.

Let O (fig. 11) be the centre of the projection, A and B the two projected poles, and OA the perpendicular on AB. Place the protractor along OA with its zero at O and read on OT, upper row of numbers, the angle corresponding to OA, divide by two and so find C. By construction 4, page 97, find C', a point which in the stereographic projection would be diametrically opposite to C. Join $C'B$. Through O draw OB' parallel to AB. Now place the protractor along OB' and read off on the upper scale of OT the position of B'. The reading so obtained is the angle between the faces projected at A and B.

Historical Appendix.

The stereographic and gnomonic projections of the sphere have proved such valuable aids in the study of crystallography that nearly every textbook of this subject deals with their construction in more or less detail. Naturally, however, the treatment in such works is concerned more with the practical application of these projections to the problems of crystallography than with their general properties, and it is thought, therefore, that a few notes of a historical character may not be devoid of interest.

Our knowledge of the *stereographic projection* we owe to Ptolemy, and it seems probable that he in his turn was indebted to the works of his great predecessor Hipparchus, whose writings on this subject are, however, no longer extant. From Ptolemy the knowledge of the projection appears to have passed to the Arabs, and was by them handed down to the mathematicians of the middle ages. As a convenient and easily constructed method of projecting the sphere on a plane it found wide application under the name of planisphaerium in the manufacture of astrolabes, and received much attention at the hands of a numerous band of writers. Among these the only authors who merit our attention are Jordanus, whose 'Planisphaerium' was written at the beginning of the thirteenth century, and Fredericus Commandinus,[1] to whom we owe the first satisfactory Latin translation of Ptolemy's work to appear in print. Accompanied by a commentary and by the 'Planisphaerium' of Jordanus, it was printed by Aldus and appeared in Venice about the middle of the sixteenth century. The end of the same century saw the publication of the monumental work of C. Clavius [2] on the astrolabe. This closely printed octavo of 759 pages is very rare, but the work was reissued in the collected edition of the mathematical writings of Clavius which appeared at Maintz at the beginning of the seventeenth century.[3]

The next work of originality and merit is that of Adrianus Metius.[4] In his 'Primum Mobile' he describes the construction of the stereographic projection, and points out how it may be used for the graphical solution of spherical triangles. His work is of special interest to us, because he was evidently acquainted with the properties of the projection on which depend the methods of finding the centres of great circles and of small vertical circles employed in the protractor described above. In the absence of formal proofs this may be concluded from the fact that he prints tables giving the lengths of the radii of these circles. The tabulated number in the case of great circles is the secant of the angle of elevation of the great circle above the plane of the primitive, and in the case of small circles, the tangent of the angular radius of the small circle. In the tables the numbers are given for every degree and to six places. To Metius we owe also a stereographic net or projection of meridians and parallels on the plane of a meridian. The net which accompanies his book was engraved by Guilielmus Blaeuw and bears date 1624. It is 29·2 centi-

[1] 'Ptolemaei Planisphaerium.' Venetiis, 1558.

[2] 'Christophori Clavii Bambergensis e Societate Iesu Astrolabium.' Romae, 1593.

[3] 'Christophori Clavii Bambergensis e Societate Iesu Operum Mathematicorum Tomus Tertius.' Moguntiae, 1611.

[4] Adrianus Metius, 'Primum Mobile.' Editio nova. Amsterdami, 1633.

metres in diameter and gives the meridians and parallels to every degree. For accuracy and delicacy of execution it will bear comparison with good modern work. It was used by Metius for the working out of astronomical problems and for the solution of spherical triangles. It is interesting to notice that Metius appears to have experienced difficulty in drawing circles of great radii, and it would seem that he contented himself with the approximation to a flat circular arc afforded by a portion of an ellipse, for he figures and describes an instrument for drawing such arcs identical in principle with the elliptic trammel in use at the present day. The only other writer of this period to whom reference need here be made is Franciscus Aguilonius [1] to whom is due the introduction of the name 'stereographic' for this projection.

In the eighteenth century the use of the globes and the projection of the sphere were favourite subjects with mathematical writers in this country, and the works of J. Harris,[2] Jo. Wilson,[3] and Charles Leadbetter [4] may still be consulted with advantage. As a knowledge of the use of the globes ceased to be an essential part of a liberal education the demand for such treatises gradually declined. Among the last and best we find a work by W. Emerson [5] in which the properties of the orthographic, stereographic, and gnomonic projections are clearly discussed. A German work of merit belonging to the same period is that of Georg Simon Klügel.[6] In recent times a formal treatise on the stereographic projection has been published by E. Reusch,[7] and sections dealing with its general properties are to be found in the textbooks devoted to map-projection, as for instance those of A. Germain,[8] G. J. Morrison,[9] C. F. Close,[10] and K. Zöppritz.[11]

[1] 'Francisci Aguilonii e Societate Iesu Opticorum Libri Sex.' Antverpiae, 1613. (See Book vi, p. 573.)

[2] J. Harris, 'Elements of Plain and Spherical Trigonometry.' London, 1706.

[3] J. Wilson, 'Trigonometry with the Doctrine of the Sphere.' 2nd edit. Edinburgh, 1724.

[4] C. Leadbetter, 'A Compleat System of Astronomy.' London, 1728.

[5] W. Emerson, 'The Projection of the Sphere.' 2nd edit., London, 1769.

[6] G. S. Klügel, 'Geometrische Entwickelung der Eigenschaften der stereographischen Projection.' Berlin und Stettin (Friedrich Nicolai), 1788.

[7] E. Reusch, 'Die stereographische Projection.' Leipzig, 1881.

[8] A. Germain, 'Traité des Projections.' Paris [1866?].

[9] G. J. Morrison, 'Maps, their Uses and Construction.' 2nd edit., London, 1902.

[10] C. F. Close, 'A Sketch of the Subject of Map Projections.' London, 1901.

[11] K. Zöppritz, 'Leitfaden der Kartenentwurfslehre.' New edition by A. Bludau, Leipzig, 1899.

As is well known, the special value of the stereographic projection depends in the first place on the property that any circle on the sphere is projected either as a circle or as a straight line, and secondly, on its angular truth. How far these properties in their general form were known to Ptolemy is difficult to discover; in all probability he was unaware of the second and acquainted with the first in special cases only. That circles are projected as straight lines or circles was known to Jordanus, and was proved for all possible cases and at great length by Clavius in the work referred to above.

The history of the second property is involved in considerable obscurity. The credit of its discovery has been freely attributed to Charles Leadbetter. A proof of the property is certainly contained in his 'Compleat System of Astronomy' and we read in the 'Dictionary of National Biography':—'He gave in this work perhaps the earliest demonstration of a well-known property of stereographic projection.' The ultimate authority for this ascription appears to be J. B. Delambre, who in his 'Histoire de l'Astronomie au dix-huitième Siècle', Paris, 1827, p. 88, makes the following reference to the 'Compleat System':—'La seule chose qui m'ait paru digne d'être extraite de ces deux volumes, est la démonstration du second théorème général de la projection stéréographique, c'est-à-dire de l'égalité des angles sur la sphère et sur la projection. Cette démonstration est la plus ancienne que je connaisse; mais elle ne paraît ni la plus claire ni la plus complète. Essayons de l'éclaircir et de la compléter.'

Leadbetter is not, however, the only Englishman to whom this credit has been assigned, for in the 'Dictionary of National Biography', under Robertson, John (1712–1776), we find:—'He is said to have been the first to discover the theorem that in stereographic projection the angle between two circles on the sphere equals the angle between the two circles on projection.' This time the authority is Chasles, 'Aperçu Historique . . . des Méthodes en Géométrie,' Bruxelles, 1837, p. 517. Referring to this property he says:—'Ce beau théorème n'a pas été aperçu par Ptolémée ni par Jordan. L'ouvrage le plus ancien, à la connaissance de M. Delambre, où il se trouve, est le traité de navigation de Robertson (1754). (Voir Traité d'astronomie, t. iii.)' Turning now to Delambre's 'Astronomie Théorique et Pratique', Paris, 1814, vol. iii, p. 674, we find the following passage:—'La première propriété paraît n'avoir pas été ignorée des Grecs, quoiqu'elle ne soit expressément mentionnée ni par Ptolémée, ni par Synésius, dans sa Lettre sur l'Astrolabe. La seconde paraît d'une date beaucoup moins ancienne; je l'ai vainement cherchée dans le gros Traité de Clavius, dans celui de Stoflérinus et dans Bion. Elle est énoncée dans le Dictionnaire de Mathématiques de Saverien, Paris, 1753, et démontrée par Robertson, dans ses Élémens de Navigation (1754).'

That Robertson was not the originator of the proof was, however, recognized by Delambre himself, for in the interval which elapsed between the publication of his 'Astronomie' in 1814 and his death in 1822 he had become acquainted, as we have seen above, with the writings of Leadbetter. But, although he was unable to find an earlier reference to the theorem, he was not disposed to attribute the credit of its discovery unreservedly to Leadbetter, for in his 'Histoire de l'Astronomie au dix-huitième Siècle', p. 90, he says:—'Nous avons la preuve que la seconde propriété de la projection stéréographique était connue dès l'an 1728, et que la date est probablement plus ancienne encore; car nous ne croyons pas que Leadbetter en soit l'auteur. Il n'y a pas non plus grande apparence qu'on soit parvenu à ce théorème par les raisonnemens de cet astronome.' As a matter of fact, Delambre was perfectly justified in maintaining this cautious attitude, for a proof identical with Leadbetter's is to be found in the works of Wilson (1724) and of Harris (1706) cited above, and it was evidently common knowledge at the time. It seems, indeed, highly probable that all these writers were indebted either directly or indirectly to an important paper published by Edmund Halley [1] in the Transactions of the Royal Society in 1695–1697.

This paper, which is of considerable interest in the history of navigation, is entitled 'An Easie Demonstration of the Analogy of the Logarithmick Tangents to the Meridian Line or sum of the Secants: with various Methods for computing the same to the utmost Exactness'. Halley says:—'For this Demonstration it is requisite to premise these four *Lemmata* . . . Lem. II. In the *Stereographick Projection*, the Angles, under which the Circles intersect each other, are in all cases equal to the Spherical Angles they represent: Which is perhaps as valuable a property of this *Projection*, as that of all the Circles of the Sphere thereon appearing Circles: but this not being vulgarly known, must not be assumed without a *Demonstration*.' The demonstration Halley gives is that found in the works already quoted and need not be reproduced here. At its conclusion he says:—'This *Lemma* I lately received from Mr. *Ab. de Moivre*, though I since understand from Dr.

[1] Phil. Trans., 1695–1697, vol. xix, No. 219, p. 202.

Hook that he long ago produced the same thing before the *Society*. However the demonstration and the rest of the discourse is my own.' All efforts to trace back the theorem to a date antecedent to Halley's paper have so far proved unsuccessful. To apportion the credit of its discovery between Halley, de Moivre, and Hook, is a delicate task, but we shall probably hardly do wrong if we assign to Halley the lion's share.

If we consider now the application of this projection to the special problems of crystallography, we find that it has been freely employed in this study ever since F. E. Neumann [1] drew the attention of crystallographers to its advantages in the first quarter of last century. The work of W. H. Miller [2] in this direction and the beautiful stereograms drawn by A. Des Cloizeaux [3] are well known.

Quite lately the method has received further development at the hands of E. S. Fedorov [4] and G. Wulff,[5] who share the credit of the introduction of the flexible ruler so useful in drawing arcs of circles of very long radii. Both these authors have, moreover, published stereographic nets for a primitive of 10 centimetres radius, and both have pointed out how the triangular compass may be used with advantage in making measurements. A smaller net for a primitive of 5 centimetres radius was issued by Benno Hecht [6] in 1893, and an extension of this beyond the boundary of the primitive has recently been published by E. Sommerfeldt [7] and employed by him in the construction of crystal figures. This projection has also been the subject of an important paper by V. Goldschmidt,[8] in which are set forth methods of dealing with the data furnished by the two-circle goniometer.

It is, however, to the writings of the late S. L. Penfield [9] that we must turn for the most exhaustive exposition of the various constructions employed in solving graphically the problems presented by crystals. To him we owe stereographic nets printed on transparent celluloid for use in measuring completed drawings as well as the introduction of

[1] F. E. Neumann, 'Beiträge zur Krystallonomie.' Berlin and Posen, 1823.

[2] W. H. Miller, 'A Treatise on Crystallography,' 1839. 'Mineralogy,' 1852.

[3] A. Des Cloizeaux, 'Manuel de Minéralogie,' vol. i, 1862, vol. ii, 1874–1893.

[4] E. S. Fedorov, Zeits. Kryst. Min., 1893, vol. xxi, p. 617, and 1903, vol. xxxvii, p. 138.

[5] G. Wulff, ibid., 1893, vol. xxi, p. 253, and 1902, vol. xxxvi, p. 14.

[6] B. Hecht, 'Anleitung zur Krystallberechnung.' Leipzig, 1893.

[7] E. Sommerfeldt, Zeits. Kryst. Min., 1906, vol. xli, p. 164.

[8] V. Goldschmidt, ibid., 1899, vol. xxx, pp. 260–271.

[9] S. L. Penfield, Amer. Journ. Sci., 1901, ser. 4, vol. xi, pp. 1–24 and 115–144; see also vol. xiii, pp. 245–275 and 347–376, vol. xiv, pp. 249–284.

sheets of paper specially designed for the construction of stereograms. On these sheets are printed circles of 7 centimetres radius divided into degrees, together with four scales. Of these, the first two give the radii of great and small circles respectively, while the third is a scale of stereographically projected degrees extending from 0° to 90° on one side of the centre, and on the other beyond the limits of the primitive up to 145°. The fourth scale is one of equal parts where each division is a one-hundredth part of the radius. These scales are based on the same principles as have been utilized above, and if the special sheets are not at hand all Penfield's constructions can be carried out with ease by means of the stereographic protractor. Indeed, a protractor graduated for a radius of 7 centimetres will be found more convenient for use with Penfield's circle than the scales printed on the sheets, as it can be placed on the drawing and points pricked off direct.

In concluding this section we may mention that a paper by J. G. Goodchild,[1] entitled 'Simpler methods in crystallography', contains some useful hints on the construction of projections, while those interested in the history of the subject will find much information in A. von Braunmühl's[2] 'History of Trigonometry' and in a paper by S. Haller[3] in which the methods of Clavius are succinctly set forth.

Gnomonic projection.—This projection has a much shorter authentic history than that we have just discussed, for though Ptolemy has been credited with a knowledge of it by Delambre, the grounds for this attribution appear to be somewhat slender, and we must seek the earliest account of it in the 'Prospectiva nova Coelestis' of Christophorus Grienberger published at Rome in 1612. It was first used in the construction of star maps, a purpose for which it is still employed. A celestial atlas on this projection was issued at Paris in 1674 by Ignatius Gaston Pardies, and reduced copies of these maps are to be found in the 'Cosmography' of Sir Jonas Moore.[4] A good account of the essential properties of the projection is given by W. Emerson,[5] and interesting information as to its early history may be found in the tract written by A. de Morgan[6] to

[1] J. G. Goodchild, Proc. Roy. Phys. Soc., Edinburgh, 1897–1901, vol. xiv, pp. 323–359 and 403–434.

[2] A. von Braunmühl, 'Vorlesungen über Geschichte der Trigonometrie.' Leipzig, 1900.

[3] S. Haller, 'Bibliotheca Mathematica.' 1899, vol. xiii, p. 71.

[4] J. Moore, 'A New Systeme of the Mathematicks.' London, 1681. (See 'Cosmography,' p. 162.)

[5] W. Emerson, loc. cit.

[6] A. de Morgan, 'An Explanation of the Gnomonic Projection of the Sphere.' London, 1836.

accompany some beautiful maps, terrestrial as well as celestial, issued by the Society for the Diffusion of Useful Knowledge, and based on a sphere of reference of 5 inches radius. At the present day charts on this projection are issued by the American Admiralty.

The applications of the gnomonic projection to crystallography have been so admirably dealt with by V. Goldschmidt [1] that we need not do more than call attention here to a few recent publications on the subject. In this country H. A. Miers,[2] G. F. H. Smith,[3] J. W. Evans,[4] and H. Hilton,[5] have all made important contributions to our knowledge. To the last we owe a gnomonic net, which may be employed in the same way as the stereographic nets already described, while G. F. H. Smith has published a table, which will be found to facilitate the plotting of crystal measurements made with the two-circle goniometer. On the continent, E. S. Fedorov has written much on the subject, and we owe to him several ingenious graphical methods, one of which was given above.

Orthographic projection.—By dropping perpendiculars on the equatorial plane from points on the sphere, we obtain the orthographic projection, a method of representation which shares the antiquity of the stereographic projection, and one which, under the name of the *analemma*, was well known to the mathematicians of the middle ages. Employed by them in the solution of astronomical problems, its use in modern times has been almost entirely confined to the construction of maps of the moon. It is ill adapted to the purposes of crystallography, but it may nevertheless claim brief mention here, for F. E. Wright [6] has recently pointed out that it is suited to the delineation of the interference-figures afforded by crystals when viewed in the polarizing microscope, and he has published an orthographic net. Such a net has also been prepared by the authorities of the Stonyhurst Observatory. It is 6 inches (= 15·24 centimetres) in diameter, and though designed to assist in the study of sun-spots, it might quite well find application in the manner suggested by Wright.

[1] V. Goldschmidt, 'Ueber Projection und graphische Krystallberechnung.' Berlin, 1887. See also, Zeits. Kryst. Min., 1890, vol. xvii, p. 97, and 1892, vol. xx, p. 143.

[2] H. A. Miers, Min. Mag., 1887, vol. vii, p. 145.

[3] G. F. H. Smith, Min. Mag., 1903, vol. xiii, pp. 309–321.

[4] J. W. Evans, Min. Mag., 1906, vol. xiv, p. 149.

[5] H. Hilton, Min. Mag., 1904, vol. xiv, pp. 18–20, 1905, vol. xiv, 99–103, 104–108.

[6] F. E. Wright, Amer. Journ. Sci., 1907, ser. 4, vol. xxiv, p. 321.

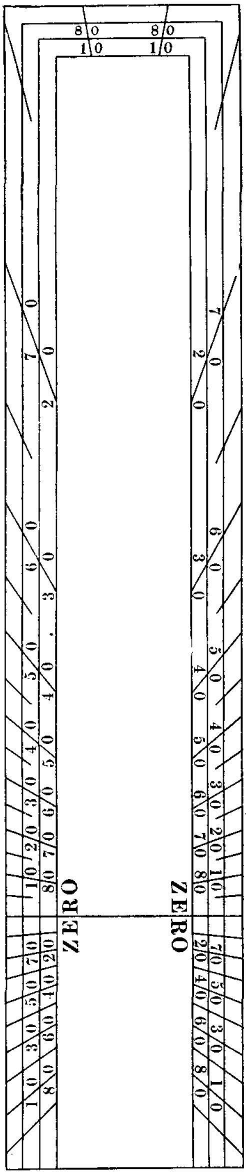


Fig. 1.

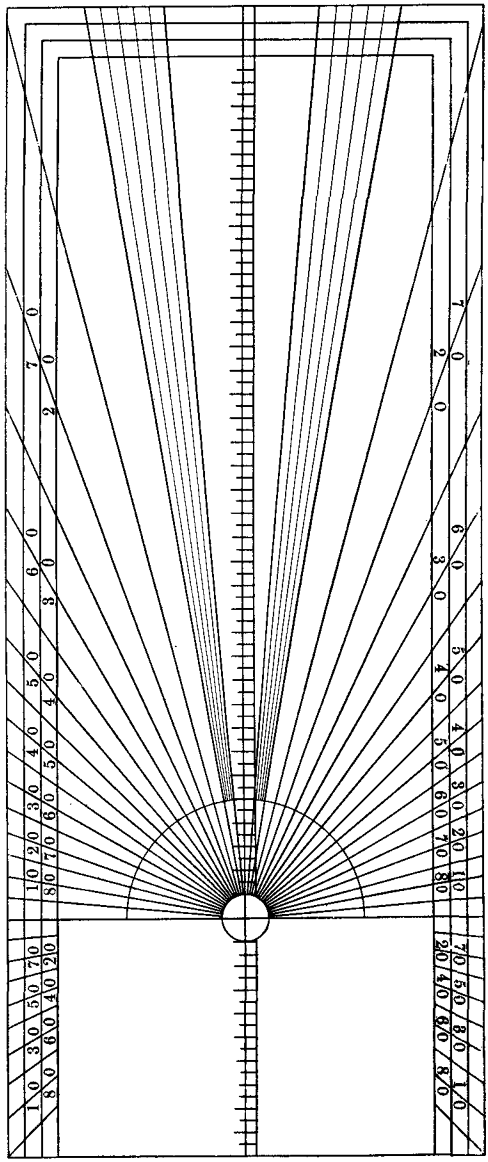
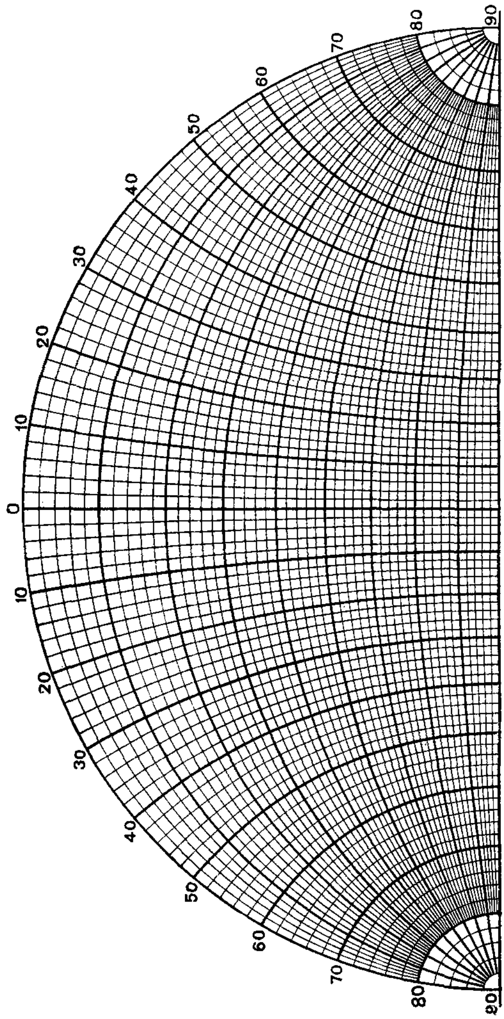


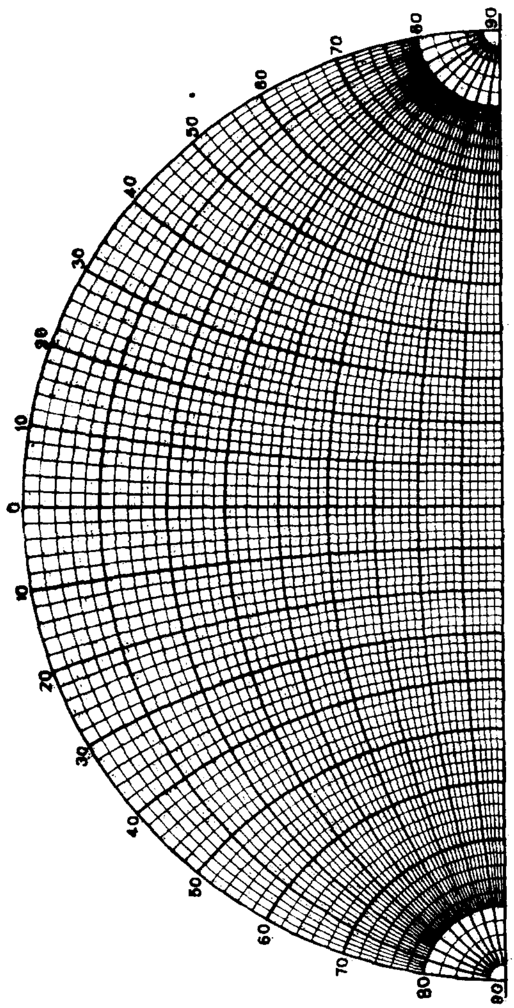
Fig. 2.

A. HUTCHINSON: STEREOGRAPHIC AND GNOMONIC PROTRACTORS.

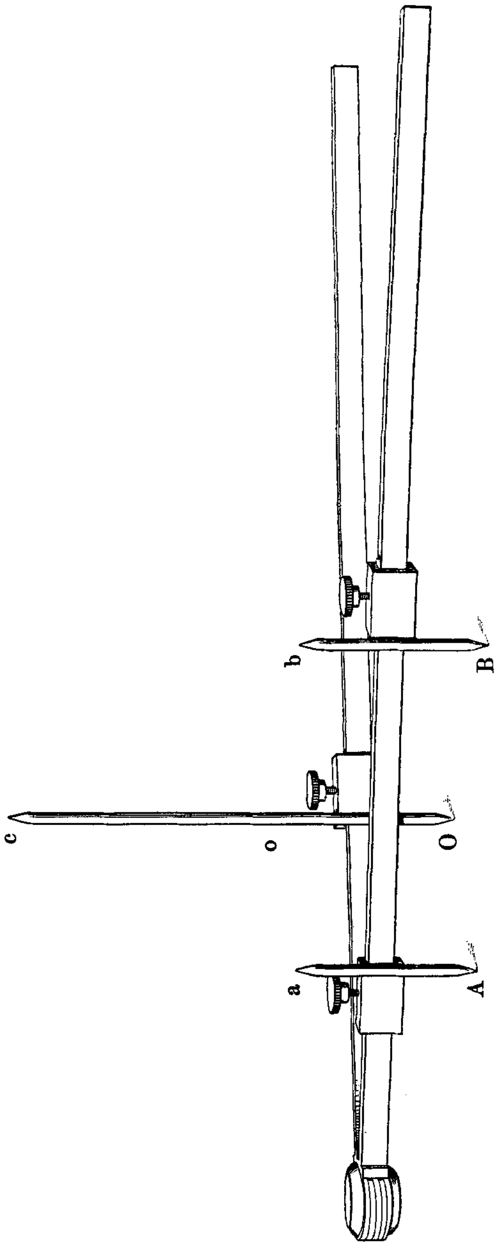
($\frac{1}{2}$ actual size. The originals are graduated to show degrees, but for the sake of clearness the finer divisions are omitted in the figures.)



A. HUTCHINSON: STEREOGRAPHIC NET.



A. HUTCHINSON: STEREOGRAPHIC NET.



A. HUTCHINSON: TRIANGULAR COMPASS.

($\frac{1}{2}$ actual size.)