The use of the orthographic projection in crystallography.

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§ 1. THE geometrical representation of the faces and edges of a crystal is obtained (i) by drawing lines through a fixed point O perpendicular to each crystal-face meeting a fixed sphere with centre O in a point (*face-pole*) representing that face, (ii) by drawing lines through O parallel to each crystal-edge meeting the sphere in a point (*edge-pole*) representing that edge. This representation is inconvenient as being in three dimensions, and therefore it is customary to map the sphere on a plane. This is usually done by means of the stereographic or gnomonic projection. But the actual drawing of the crystal is made on a different projection, namely the orthographic (or orthogonal). There are therefore many advantages in mapping the sphere on the same projection; any point p of the sphere being mapped on a fixed plane Π through O by the foot P of the perpendicular pP from p on the plane Π. The map gives the appearance of the sphere as seen by an eye at a considerable distance situated in the normal through O to Π. We shall call the intersection of Π with the sphere the *fundamental circle* of the projection.[1]

The great advantage of this method consists in the fact that the line joining O to the orthographic projection of an edge-pole is parallel to the corresponding edge in the final drawing of the crystal. The finding of the directions of these edges is a necessary preliminary to the execution of the drawing.

The main disadvantages of the orthographic projection are (i) that it is not so easy to draw as the stereographic or gnomonic projection, for the projection of a great circle is not a circle or straight line but an ellipse

[1] We denote the projections of $p, q, \ldots$ by $P, Q, \ldots$ throughout. If P lies on the fundamental circle, p coincides with P.

with a diameter of the fundamental circle as major axis, and (ii) that there is undue compression near the circumference of the fundamental circle. The objection (ii) is less important than appears at first sight, for the resulting error in plotting the position of an edge-pole makes but little difference to the final drawing of the crystal. We shall consider objection (i) later on in more detail.

§ 2. The easiest way to draw the projection is by means of the *orthographic net*. This is an orthographic projection of the great circles whose planes pass through a diameter λ of the sphere (meridians) and of the small circles having λ as axis (lines of latitude) on to a plane through λ. The projections of the meridians are ellipses with λ as a common major axis. Their minor axes lie along a line μ perpendicular to λ. The projections of the lines of latitude are straight lines perpendicular to λ.

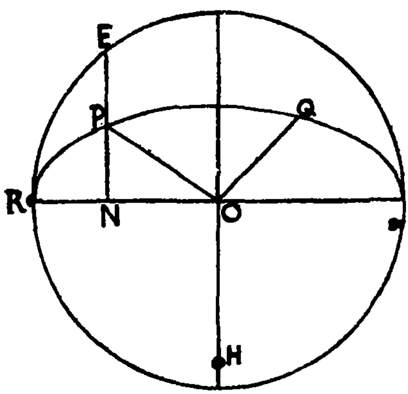

FIG. 1.

The use of the net is exactly similar to that of the well-known gnomonic and stereographic nets; and by their aid any one of the three projections is drawn with equal ease.

The projection is drawn on tracing-paper, the centre *O* of the drawing being always over the centre of the net.

For instance, suppose we have the projections *P*, *Q* of two face-poles *p*, *q* (fig. 1). Then to get the projection *H* of the pole *h* of the edge in which the two faces meet, turn the diagram about *O* until *P* and *Q* lie on the same meridian of the net, and measure off along the line μ an angular distance of 90° from this meridian. Thus *H* is obtained.

Gnomonic, stereographic, and orthographic nets were drawn by F. E. Wright and appended to his work on 'The methods of petrographicmicroscopic research' (Washington, The Carnegie Institution, 1911). He does not apply these nets to crystal-measurement. Further details as to the use of nets, with references, will be found in papers by the author and Dr. Hutchinson in volumes xiv and xv respectively of this Magazine.

The cross-ratio of four co-zonal faces with poles p, q, r, s is equal to the cross-ratio of the pencil $O(PQRS)$, where O is the centre of the projection. It may be obtained graphically, when the projection has been drawn, by measuring the intercepts made by the rays of this pencil on a line parallel to any one of them.

§ 3. Suppose now that we have an orthographic projection on the plane Π of the face- and edge-poles: and that we require the projection on Π when the sphere and the crystal, which is supposed rigidly attached to the sphere, is rotated about an axis through O.[1] There will be no loss of generality in supposing the rotation to take place about an axis in Π. For any rotation may be shown to be equivalent to a rotation about an axis in Π followed by a rotation about the normal to Π; and this second rotation does not alter the drawing of the crystal. Suppose then the rotation to be through an angle θ about an axis in Π. We place this axis along the diameter λ of the net and move each pole of the diagram along its line of latitude (perpendicular to λ) through an angular distance θ, as given by the meridians of the net. We have then the new projection required.

It is important to notice that, if we draw the pictures of the crystal in the two positions, the line joining the two positions of a given vertex of the crystal is perpendicular to the axis of rotation λ. Hence, if one drawing has been obtained, the other is immediately derived. For we know the directions of the edges of the crystal in the other drawing, and lines perpendicular to λ along which its vertices lie. For the first drawing we may with advantage choose Π perpendicular to an axis of symmetry, or perpendicular to some important zone of the crystal, as recommended also by other authors.

§ 4. We now show how to draw the 'axial cross' of a crystal. Let the crystallographic axes Oa, Ob, Oc meet the sphere in a, b, c. The projections A, B, C of a, b, c are at once drawn by the aid of the net in the case in which Ob and Oc lie in the plane Π (fig. 2). Let OL be the line in Π perpendicular to OC. The axial cross is now usually drawn by rotating the trihedron Oa, Ob, Oc about OC through an angle θ and then about OL through an angle ϕ. These rotations are equivalent to a rotation through an angle ψ about a line OK in Π followed by a rotation through

[1] It comes to the same, if we suppose the sphere kept fixed and Π altered.

an angle ϵ about the normal to Π, where $\tan KOC = \tan\phi \,.\, \operatorname{cosec}\theta$, $\tan\frac{1}{2}\epsilon = \tan\frac{1}{2}\theta \,.\, \tan\frac{1}{2}\phi$, and $\cos\frac{1}{2}\psi \,.\, \cos\frac{1}{2}\epsilon = \cos\frac{1}{2}\theta \,.\, \cos\frac{1}{2}\phi$.

In this all rotations are in the clockwise direction to an observer looking along the axis of rotation towards O.[1]

The final projection of the ends of the crystallographic axes is now found by making OK coincide with the diameter λ of the net, and moving A, B, C perpendicular to λ through an angle ψ as given by the meridians of the net. The projection of the crystallographic axes is found by joining O to the new positions of A, B, C. The 'axial cross' is obtained by multiplying OA, OB, OC by the axial ratios.

In fig. 2 we show the case in which the angles between the crystallographic axes are 100°, 108°, 94°. The tail and point of the arrows are the original and final positions of A, B, C respectively.

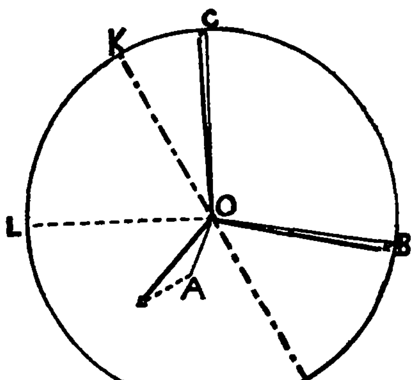

FIG. 2.

§ 5. If an orthographic net is not to hand, we may replace it by the following device, which also avoids the use of tracing-paper. We shall suppose the fundamental circle to be of 10 centimetres radius. On paper ruled in millimetre squares draw $XY =$ 10 cm., and XZ perpendicular to it, along lines on the squared paper. Graduate XY as in fig. 3; the division representing an angle θ being $10 \,.\, \sin\theta$ centimetres from X. Join the division marks to Z.

Suppose the crystal to be measured on a two-circle goniometer. Take the plane Π parallel to the central face (perpendicular to the principal

[1] e.g., if $\theta = \tan^{-1}\frac{1}{3} = 18° \, 26'$, $\phi = \tan^{-1}\frac{1}{6} = 9° \, 28'$, then $COK = 27° \, 48'$, $\epsilon = 1° \, 32'$, $\psi = 20° \, 38'$. These values are taken in fig. 2.

zone) of the goniometric readings. If p is the pole of any face at an angular distance ρ from the central face, the distance OP in the projection is $10 . \sin \rho$ centimetres. Hence the projection of the face-poles is at once drawn.

Now suppose we have two face-poles p, q, and that h is the edge-pole of the edge in which the two faces with poles p, q meet. We know P, Q, and require to find H (fig. 1).

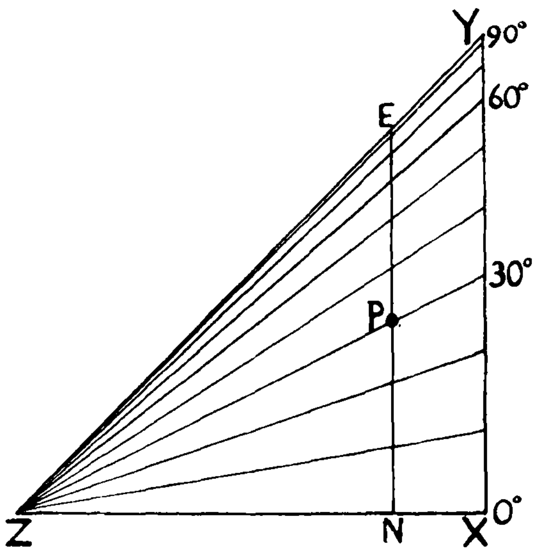

FIG. 3.

Suppose that the great circle pq meets the fundamental circle in r (coinciding with R). If the position of R has not been already obtained, we may calculate this position from the formula

$$\tan \sigma . \sin (op - oq) = \tan \tau . \sin (op + oq),$$ [1]

where $POR = \sigma - \tau$, $QOR = \sigma + \tau$; so that $\tau = \frac{1}{2} POQ$. We know τ, op, oq from the goniometric readings. The position of H is at once obtainable, if we know the angle θ between the great circles pqr and or. It may be calculated from

$$\tan \theta = \tan op . \sin (\sigma - \tau),$$

or obtained graphically as follows. Place a millimetre ruler along EPN in fig. 1 perpendicular to OR. Place the line EPN in fig. 3 with E on

[1] o is the point on the sphere whose projection is O, so that oO is perpendicular to Π.

ZY, N on ZX, and EN perpendicular to ZX. The value of θ is that given at the right-hand end of the line ZP. For the ratio PN/EN is the same for all points p in the zone qr.

Of course θ may be at once read off from the ruler and the squared paper of fig. 3 without actually drawing EN in either fig. 1 or 3. In fact, it is not necessary to have the two figures on the same scale. I have found it convenient to draw fig. 3 on tenth-inch paper with $XY = 10$ inches. Such a diagram may be drawn graduated to every other degree in half an hour or less. A strip of millimetre squared paper serves excellently for the ruler applied to the projection in fig. 1.

Suppose now we have the projection P of any point p on the sphere (fig. 1), and require the new position of P after the sphere has been rotated about any line of Π (say OR) through a known angle ϵ. Place the line EPN in fig. 3 as just described. Move P along the line EN perpendicular to XZ through an angle ϵ as given by the lines through Z. This gives the new position of P, which can be now inserted in fig. 1. As before, we do not actually draw the line EPN, but read off the new position of P immediately on the ruler and squared paper of fig. 3.

To obtain the 'axial cross' we require the positions of A, B, C in fig. 2. We know the angles between the crystallographic axes Oa, Ob, Oc. Hence B and C are at once given. To get A, we note that, since the angle cOa is known, A lies on a known straight line perpendicular to OC. Similarly A lies on a known straight line perpendicular to OB, and thence A is obtained. The new positions of A, B, C after rotating about OK are obtained by the process just described.

§ 6. We illustrate our discussion by showing the drawing of a simple orthorhombic crystal. The observed goniometric readings on the two-circle goniometer are:

face	b	m	m_1	r	r_1
ϕ (longitude)	0°	0°	180°	39°	141°
ρ (distance from central face)	90	53	53	90	90

and the faces parallel to these also exist. We shall denote the projections of the edge-poles by numbers, thus $bmm_1 = 1$, $brr_1 = 2$, $mr = 3$, $mr_1 = 4$, $m_1r = 5$, $m_1r_1 = 6$.

In fig. 4 (i) we show the orthographic projection of the face-poles on Π when the central face (for which $\rho = 0$) is parallel to Π; and the projection of the edge-poles when the central face is parallel to Π and after the crystal has been rotated through 20° about the line λ. The tail of each arrow is the projection of an edge-pole before the rotation, and the

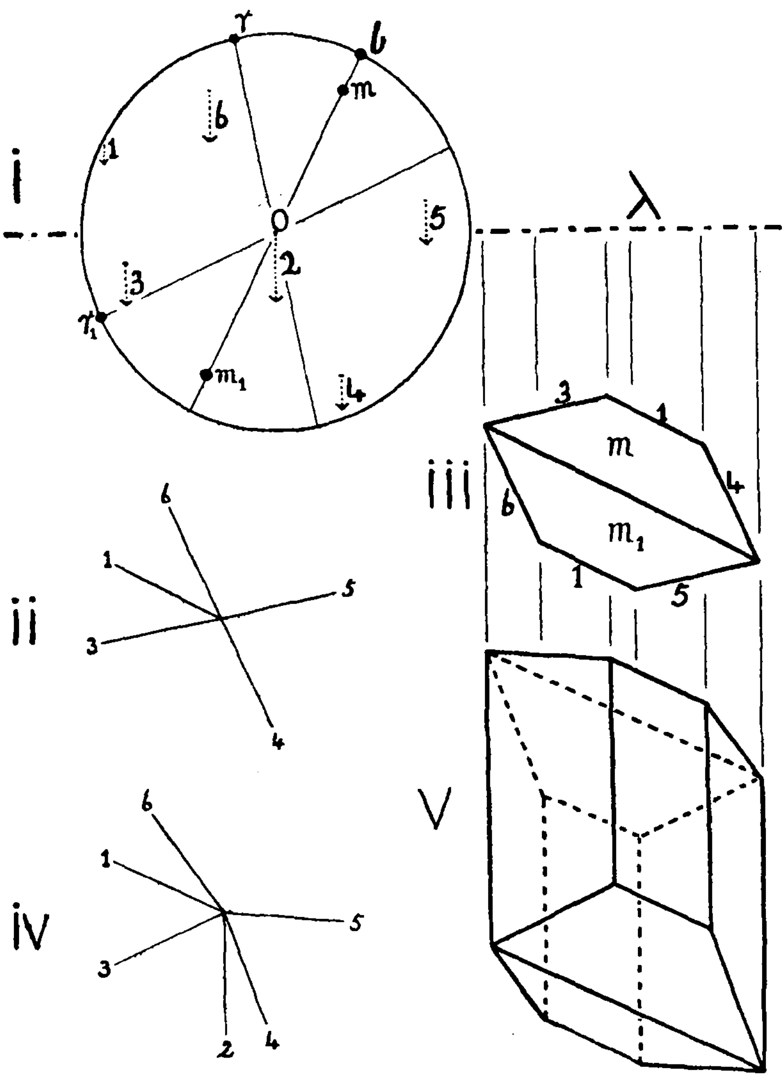

FIG. 4.

point of the arrow the projection after rotation. Of course each arrow is perpendicular to λ. In (ii) are shown the directions of the lines joining O to the projections of the edge-poles before rotation (the tails of the arrows). In (iii) we show the corresponding picture of the crystal. We note that the edges in (iii) are parallel to the lines in (ii); and then symmetry and a knowledge of the crystal's habit enable us to complete the drawing (iii). In (iv) are shown the directions of the lines joining O to the edge-poles after rotation (the points of the arrows). In (v) we show the corresponding picture of the crystal. The part of the crystal shown in (iii) is at the bottom of (v). We note that the edges in (v) are parallel to the lines in (iv), and that the lines joining corresponding vertices of (iii) and (v) are perpendicular to the line λ. This enables us to complete the drawing (v); the ratio of the lengths of edges 1 and 2 being given by the crystal's habit.

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