The numerical relation between the zones and faces of a polyhedron.

By E. S. Fedorov (E. C. Федоровъ),
Professor of Crystallography in the Institute of Mines, Petrograd.

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In Professor G. Cesàro's note[1] 'Sur une propriété numérique de l'ensemble des axes de symétrie situés dans les plans de symétrie d'un polyèdre' is given the formula ((1) on p. 174)

$$X^2 = X + (n-1)\Sigma N_1 + (p-1)\Sigma P_1 + (q-1)\Sigma Q_1 + \ldots$$

or

$$(X-1)X = (n-1)\Sigma N_1 + (p-1)\Sigma P_1 + (q-1)\Sigma Q_1 + \ldots,$$

in which '$N.\Lambda^n$, $P.\Lambda^p$, $Q.\Lambda^q$,... est l'ensemble des axes de symétrie qu'un polyèdre contient dans ses plans de symétrie'. From the last-named condition it might appear that this formula is only applicable in the particular case when the axes of symmetry are the intersection-directions of planes of symmetry. It has, however, a wider and more varied application, and moreover is identical with the formula[2]

$$(p-1)p = 1.2f_2 + 2.3f_3 + \ldots + (n-1)nf_n + \ldots$$

which was deduced by the present author before 1880, and applied with two different significations.

If we term the product $m(m-1)$, where m is a whole number, 'the function' of this number, then the above formula in one of its significations is an expression of the theorem: *The function of a zonohedron is the sum of the functions of its faces.*

A 'zonohedron' is defined as a polyhedron whose faces lie in the primitive zones on the crystal, such, for example, as the cube or rhombic-dodecahedron. Its faces are polygons with an even number of sides in parallel pairs. The symbols $f_2 \ldots f_n$ in the above formula denote the

[1] G. Cesàro, Mineralogical Magazine, 1915, vol. xvii, pp. 173-177.

[2] E. S. Fedorov, 'Elements of the theory of figures' (Russ.), Verh. Russ.-Kais. Mineralog. Gesell. St. Petersburg, 1885, ser. 2, vol. xxi, pp. 1-277, with 18 plates. See §§ 65, 70 (formula on p. 192). Abstract in Zeits. Kryst. Min., 1893, vol. xxi, pp. 679-714 (formula on p. 689).

number of faces with n pairs of parallel edges (two parallel faces being regarded as identical), and p means the number of zones. For the cube $p = 3$ and $f_2 = 3$; introducing these values into the formula we get $2 . 3 = 1 . 2 . 3$. For the rhombic-dodecahedron we have $p = 4$ and $f_2 = 6$; thus $3 . 4 = 1 . 2 . 6$. It was on this account that the formula was termed 'zonohedral'.

In its second signification the formula implies a corresponding theorem, relating to a polyhedron with parallel faces. The theorem can be stated as follows: *The function of a paired-faced polyhedron is the sum of the functions of its zones.*

Here the formula is termed 'polar zonohedral'; and p is now the number of pairs of parallel faces, and f_n is the number of zones passing through n pairs of parallel faces. For the octahedron $p = 4$, $f_2 = 6$; thus $3 . 4 = 1 . 2 . 6$. For the rhombic-dodecahedron $p = 6$, $f_2 = 3, f_3 = 4$; thus $5 . 6 = 1 . 2 . 3 + 2 . 3 . 4$.

In neither case does the formula bear any relation to symmetry. But in the very special case, dealt with by Cesàro, in which axes of symmetry are the intersections of planes (P) of symmetry, we have a polyhedron with the number of faces $2P$; and for each zone with axes Λ^2, Λ^3, Λ^4 we have $f_2 = 2, 3, 4$ respectively. Thus in the case $3\Lambda^2$, $4\Lambda^3$, we have $P = 6$, and $5 . 6 = 1 . 2 . 3 + 2 . 3 . 4$; in the case $6\Lambda^2$, $4\Lambda^3$, $3\Lambda^4$, we have $P = 6 + 3$, and $8 . 9 = 1 . 2 . 6 + 2 . 3 . 4 + 3 . 4 . 3$; and in the case of the *regular* dodecahedron $15\Lambda^2$, $10\Lambda^3$, $6\Lambda^5$, we have $P = 15$, and $14 . 15 = 1 . 2 . 15 + 2 . 3 . 10 + 4 . 5 . 6$.