On the use of the gnomonic projection in the calculation of crystals.

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CONSIDERED from the crystallographical point of view the outstanding feature of the gnomonic projection is the fact that all great circles on the sphere, and consequently all zones on the crystal, appear on the diagram as straight lines. In order to secure its full advantages, it is necessary to take the plane of the paper at right angles to the axis of a zone, and preferably one of the principal zones of symmetry on the crystal. In such an arrangement the points on the diagram corresponding to the poles in the zone mentioned—which for convenience in reference to the diagram we may term the equatorial zone—all lie at infinity, and therefore, if the straight lines representing the zones passing through the principal poles of symmetry in the equatorial zone are drawn, the diagram takes the form of a network consisting of parallelograms (fig. 1), which in particular cases become rectangles, or even squares. Except in the monoclinic system when the pole of symmetry lies in the equatorial zone, and always in the triclinic system, the pole of the projection O coincides with a node of the network.

The linear distance separating the nodes lying on any one straight line on the diagram are very simply related to the indices of the corresponding faces. In fig. 1, for instance, we have a series of parallel straight lines, $P_1P_2P_3$, $Q_1Q_2Q_3$, $R_1R_2R_3$, representing zones passing through a pole A lying in the equatorial zone. The straight line $Opqr$ is drawn through the pole of projection at right angles to the series intersecting them successively as shown. Suppose the angular distances

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of the poles corresponding to $P_1P_2P_3$ measured from the pole A are ρ_1, ρ_2, ρ_3 respectively, and the azimuthal angle of the zone measured from the equatorial zone is ϕ, then, if a be the radius of the sphere of projection, we have $Op = a \cot \phi$, $pP_1 = a \cot \rho_1 . \operatorname{cosec} \phi$, and similarly for P_2 and P_3, whence it follows that

$$\frac{P_1P_3}{P_1P_2} = \frac{pP_3 - pP_1}{pP_2 - pP_1} = \frac{\cot \rho_3 - \cot \rho_1}{\cot \rho_2 - \cot \rho_1},$$

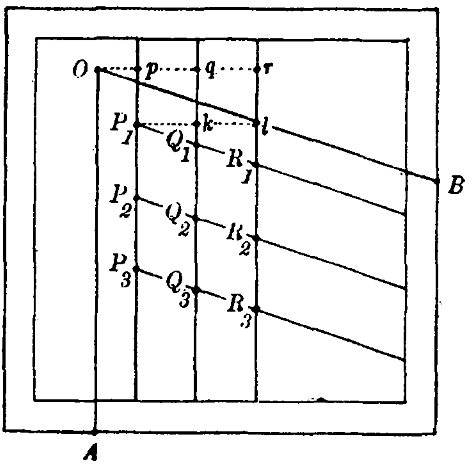

FIG. 1.—Gnomonic projection.

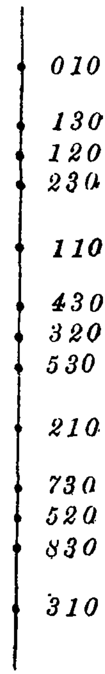

FIG. 2.—Zone in gnomonic projection.

which is equal to the ordinary anharmonic ratio subsisting between the indices of the corresponding faces. Suppose now that P_1 is (010), P_2 (110) (fig. 2), then (210) is just twice as far from (010) as (110) is, and (310) three times, and so on. Similarly, to find the point corresponding to (120), we bisect the unit (010) : (110), and that corresponding to (130), we take a third of the distance, and so on.

In the modern method of measuring crystals the position of a pole is determined by means of bi-angular co-ordinates—generally its distance measured from a fixed pole called the origin, and its azimuth measured from a fixed plane passing through the latter. To make the fullest use of the zonal properties of crystals, we must take as origin a pole corresponding to a face or at least a possible face on the crystal, and preferably one of symmetry. If the origin lies in the equatorial plane and we prepare a gnomonic projection in the manner described above, we shall in the general case have a diagram similar to the one shown in fig. 1. Suppose A be the origin, then the linear co-ordinates of P_1 on the diagram are given by $Op = a \cot \phi$; $pP_1 = a \cot \rho_1 . \operatorname{cosec} \phi$. The radius of the sphere of projection a should be taken as 10 cm., or similar simple unit.

The first of the co-ordinates given above, which depends on the azimuth only, is readily found, because the unit distance pq and the linear co-ordinate of one of the points p, q, r are two of the factors which are either taken direct from the observations or easily calculated from them, using a table of natural cotangents. When we know the co-ordinate of such a point as p we at once deduce that of any other by multiplying or dividing the unit distance by a simple number or a simple ratio. Any large error is easily guarded against if we measure the distances on the diagram. A reference to the table of cotangents gives us at once the corresponding angle.

The calculation of the angular distances presents greater difficulty, especially in the general case. If the pole of projection O corresponds to a possible face on the crystal it coincides with one of the nodes of the network, and many of the difficulties disappear; the unit distance, for instance P_1P_2, is either given directly by, or readily calculated from, the observations made in the zone passing through O and A. In the case of a triclinic crystal it might be necessary to calculate the co-ordinates of C (fig. 3 a), one of the principal poles. If the poles are determined as the intersection of zones[1]—a convenient way of working up a crystal for which a three-circle goniometer is very useful—then in the spherical triangle ABC (fig. 3 a) we know the side c and the angles A and B, and can compute in the ordinary way the other two sides, which are the distances of C from A and B, and the angle C. We have next to find the unit distance such as P_1P_2. If we use as data some of the azimuths at B, we know or can readily compute the length of Cm (fig. 3 b), and we have $Ce = Cm \operatorname{cosec} c$, the angle at the point C in the diagram being equal to the side c in the equatorial zone. Once we know one of the nodes such as P_1 in a straight line, and

[1] See Mineralogical Magazine, 1902, vol. xiii, p. 126.

the unit distance in the series of straight lines to which the latter belongs, as before we find the co-ordinates of the other nodes by adding or subtracting simple proportion of the unit, and can check the accuracy of the computation by direct measurements on the diagram. To obtain the corresponding angular values, we have to multiply the linear co-ordinates by sin ϕ, and look up in the tables the angles corresponding to the cotangents thus found. If the diagram consists of rectangles, we have from zone to zone the same linear co-ordinates in the several straight lines, and have only to multiply by the sine of a different azimuthal angle. In the general case, however, to pass from a pole such as P_1 to another such as Q_1 we shall have to add or subtract a simple proportion of kQ, which is equal to pq cot c.

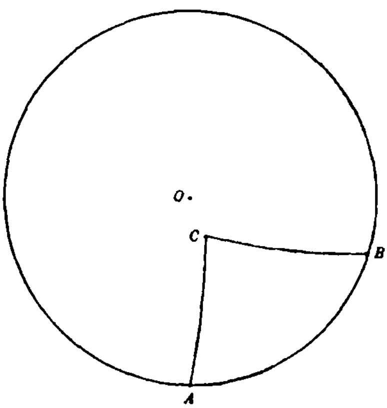

FIG. 3 a.

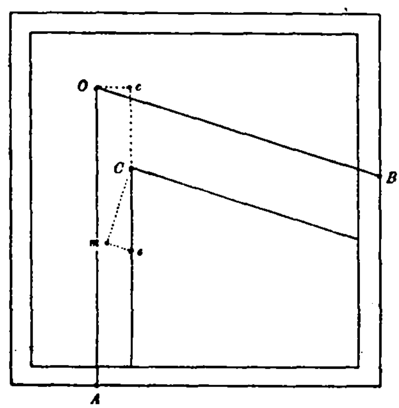

FIG. 3 b.

Stereographic and gnomonic projections of a triclinic crystal.

This method was devised to facilitate the lengthy calculations required in the case of sartorite.[1]

The only serious objection to the use of the gnomonic projection in the general discussion of the morphological characters of crystals is that from its nature it is impossible to show on it the points representing the faces whose normals are parallel to the plane of the projection, or in fact to show on a diagram of reasonable size the points representing faces whose normals are inclined to the perpendicular to the plane of the projection at angles greater than 65°. In practice the objection is not of much moment, because few faces on a crystal, with the exception of those in the equatorial zone, are so

[1] This vol., p. 259.

steeply inclined that their poles lie beyond the limit mentioned, and the equatorial zone may be depicted in the usual way on the outer edge of the diagram. Nevertheless, for some purposes it is convenient to view the whole of the poles representing the faces on a crystal, and we may accomplish this by projecting on to the faces of a cube touching the sphere of projection at, if possible, the principal poles of symmetry.

Now, when a straight line representing a zone crosses the boundary on the diagram separating the two planes of projection, its direction is in general changed, but the new direction is readily found by graphical means in the following manner. Suppose A and B (fig. 4) are the oles

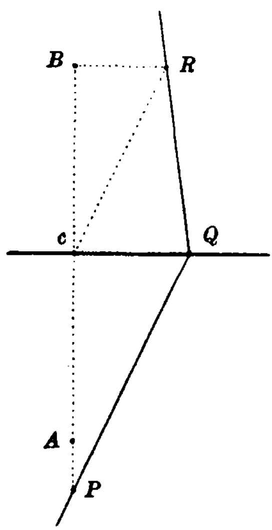

FIG. 4.

of projection on the two planes of projection. Join AB, and let it meet the boundary at c. Let PQ be some zonal line in the first plane, and let it meet AB, produced, if necessary, in P. If PQ be inclined to BA at an angle θ, the poles in which the corresponding zones intersect the equatorial zone, whose edge is perpendicular to this plane, subtend at the centre of the sphere the same angle θ. This equatorial zone appears on the second plane as a straight line through B parallel to the boundary between the two planes, and, if R is the second pole, we have $BR = a \tan \theta$, where a is the radius of the sphere. But Bc is a. Therefore the angle BcR is θ, and cR is parallel to PQ. Now R is a pole on the zone in question, and hence, if we join QR and prolong it, we have on the second plane the straight line representing that zone. The rule, therefore, for finding the new direction is very simple. Draw through the point *c* a straight line parallel to the zonal line in the first diagram. The point in which it meets the *straight* line through *B* at right angles to *Bc* represents a pole on the zone. The straight line

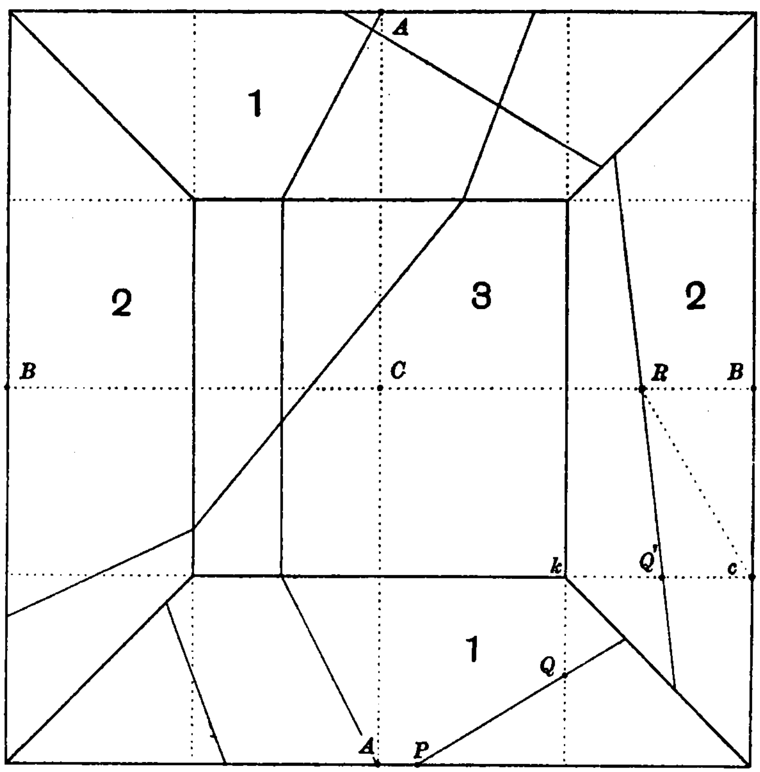

FIG. 5.—Composite gnomonic projection on five adjoining faces of a cube.

passing through this point and the point in which the zonal line in the first plane intersects the common boundary is the zonal line in the second plane.

Fig. 5 shows a composite gnomonic projection of the kind suggested. Suppose that the cube face 3 lies in the plane of the paper, and that the cube faces 1 and 2 are turned about their respective edges with 3 until they likewise lie in the plane of the paper. Since faces on the under half of the crystal may be represented by the parallel faces on the upper half, the lower halves of the diagram drawn on the side faces 1 and 2 need not be included. The latter faces, after rotation into the plane of the paper, meet only at the respective corners, but the intervening gaps may conveniently be made use of for continuing the zonal lines up to the boundaries equally inclined to both squares in the manner depicted in the figure; this method may be very useful in the case of poles lying on one or other cube face near their common edge. Of course, the poles appearing in the triangles adjoining 2 appear also on the face 3, and *vice versa*. The procedure for finding the new direction of a zonal line crossing the boundary between the cube faces 1 and 2 is only slightly different from that given above. Let us consider the zonal line $PQQ'R$ in fig. 5; then $kQ = kQ'$, and cR is drawn at right angles to PQ. It will be noticed that zonal lines passing through the pole of projection on one of the cube faces, after crossing a boundary, traverse the next face in the direction at right angles to that boundary.