The possible axes of crystal-symmetry.

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THE following is a simple general proof that, on the hypothesis that crystals have a homogeneous cellular structure, the only possible axes of symmetry are those with cyclic numbers 2, 3, 4, or 6.

According to this hypothesis, every crystal is entirely made up of minute space-partitions or cells, which (apart from those of purely surface layers) have all the same form, size, internal configuration, orientation, and external relations. Accordingly, every point in one cell has an *equivalent* point exactly corresponding to it in every other cell. From the identity of external relations it follows that any two equivalent points form part of a rectilinear line or *row* of equivalent points at equal distances from one another, and that every such row is one of a series of parallel rows of identical character; also that any two intersecting rows of equivalent points lie in a plane or *net* of equivalent points arranged in parallel rows in different directions, and that such a net forms one of a series of parallel nets with identical characters.

If now the crystal possess an axis of simple or co-directional symmetry (see this volume, p. 288), there must be a local axis of symmetry with the same cyclic number, character, and orientation traversing, in exactly the same manner, every individual cell.[1] It follows that there must be a series of nets of equivalent points at right angles to an axis of symmetry; for, if any point in the crystal-structure be taken, there will be by virtue of every local axis of symmetry (except that one, if any, which traverses the point taken) one or more equivalent points lying in a plane which is at right angles to the axis of symmetry and contains the original point; and as there are an indefinite number of local axes of symmetry parallel to one another, there must be an indefinite number of equivalent points lying in the same plane at right angles to them.

Now consider any plane at right angles to an axis of symmetry. This will be met by the local axes of symmetry in equivalent points that

[1] This would include an axis traversing a plane forming part of the cell boundary.

form a net. Let O and P be two of such equivalent points on local axes of symmetry which adjoin each other on any row in that net—that is to say, which have no other intermediate equivalent point between them. Let their distance from one another be a.

Then since O is situated on a local axis of symmetry and OP is at right angles to it (see figs. 1–5), there will be another point Q equivalent to P at the same distance a from the point O in such a position that the angle $POQ = \frac{2\pi}{n}$, where n is the cyclic number of the axis of symmetry and therefore an integer; and it will not differ in orientation.

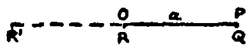
In the same manner, since the point P is on a local axis of symmetry, there will also be an equivalent point R, on the same side of OP as Q, such that $PR = OP$ or a, and the angle $OPR =$ the angle $POQ = \frac{2\pi}{n}$. Then QR will form part of a row of equivalent points, and since it is parallel to OP, and Q and R are equivalent to O and P, the row QR must be in all respects identical with the row OP, and equivalent points must occur along it at the same intervals as along OP. Accordingly as O and P are adjoining points on OP and therefore at the minimum distance a, QR must be an integral multiple of OP, or a.

Through O draw OR' parallel to PR, meeting QR at R'. Then $QR = R'R + QR' = a + 2a \cos OQR$ (or OQR^1) $= a - 2a \cos POQ = a\left(1 - 2\cos\frac{2\pi}{n}\right)$. In order that this should be an exact multiple of a, $2\cos\frac{2\pi}{n}$ must be an integer, positive, negative, or zero; $\cos\frac{2\pi}{n}$ must therefore be either an integer or half an integer. As its limiting values are $+1$ and -1, its only possible values are $+1$, $+\frac{1}{2}$, 0, $-\frac{1}{2}$, and -1. The corresponding values of $\frac{2\pi}{n}$ and n are shown by the following table:

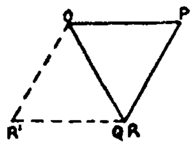
$QR = a\left(1 - 2\cos\frac{2\pi}{n}\right)$	$2\cos\frac{2\pi}{n}$	$\cos\frac{2\pi}{n}$	$\frac{2\pi}{n}$	n
$-a$	$+2$	$+1$	$\left\{\begin{matrix} 2\pi \\ 0 \end{matrix}\right.$	1 *infinity*
0	$+1$	$+\frac{1}{2}$	$\frac{\pi}{3}$	6
$+a$	0	0	$\frac{\pi}{2}$	4
$+2a$	-1	$-\frac{1}{2}$	$\frac{2\pi}{3}$	3
$+3a$	-2	-1	π	2

The value $n = 1$ signifies that a rotation through a complete circle is necessary if coincidence is to occur, or, in other words, that the axis has no special symmetry. The value $n =$ infinity implies that the cells are infinitely numerous and therefore infinitely small.

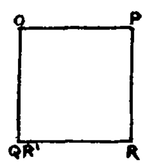
The corresponding diagrams are as follows :—

Fig. 1.— 

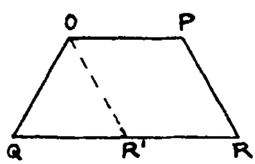
$QR = -a$; $n =$ infinity, or 1.

Fig. 2.— 

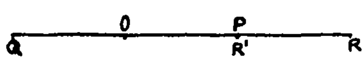
$QR = 0$; $n = 6$.

Fig. 3.— 

$QR = +a$; $n = 4$.

Fig. 4.— 

$QR = +2a$; $n = 3$.

Fig. 5.— 

$QR = +3a$; $n = 2$.

Figs. 1-5. Possible axes of crystal-symmetry.

In the case of contra-directional symmetry—in which a rotation of $\frac{2\pi}{n}$ results not in coincidence but in a reversal of uniterminal directions—similar considerations hold. The points Q and R will either be exactly equivalent to O and P or differ from them only in the orientation of their relations. In either case QR must be an integral multiple of OP and the proof already given applies.

Story-Maskelyne deduced from the law of rationality of indices the condition that $2\cos\frac{2\pi}{n}$ is rational, and then proved that, if n is an integer, the only rational values of $2\cos\frac{2\pi}{n}$ are integers (W. J. Lewis, Crystallography, 1899, p. 119).