The biaxial ray surface.

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Summary. Existing descriptions of the biaxial ray surface are, in general, misleading. A more accurate description, stressing the simplicity of the surface, is proposed and this is supported by numerous sections drawn through or near the singular points.

To the mathematician, the equation to the ray surface (Fresnel, 1827, p. 138)

$$a^2x^2(b^2-r^2)(c^2-r^2)+b^2y^2(a^2-r^2)(c^2-r^2)+c^2z^2(a^2-r^2)(b^2-r^2) = 0 \quad (1)$$

indicates one continuous surface whose configuration is indicated exactly and in every detail by the equation (e.g. Bell, 1926, pp. 267–269). To those who are less mathematically competent, including the ordinary student of crystal optics, such an equation has little meaning: students in this category must rely upon published sections and written descriptions for their impressions of the surface.

Most published sections of the biaxial ray surface tend to give one the impression that it consists of two surfaces, one totally enclosed within the other, in contact with each other at four points; published verbal descriptions do little to erase this misconception.

The methods used by different authors to describe the ray surface fall in general into three categories (purely analytical unillustrated treatments are not considered):

Analytical or non-analytical treatment in three coordinate planes, usually with illustration of these three planes; e.g. Fletcher (1892, pp. 32–35) and Born (1933, pp. 234–238).

Analytical or non-analytical treatment of three principal planes supported by three-dimensional representation showing all three planes, but with virtually no verbal description; e.g. Eskola (1946, pp. 109–110), Groth (1905, pp. 91–96; 1910, pp. 129–135), Joos (1934, pp. 357–358), Liebisch (1896, pp. 357–358), Miers (1929, pp. 159–160), Nye (1957, p. 240), Rogers and Kerr (1942, pp. 99–101), Schulz (1928, p. 472), Tunell (1933, pp. 335–336), and Wahlstrom (1949, pp. 124–128). In addition, Joos (1934, fig. 3, plate 1) provides a photograph of a well-made solid model of the ray surface depicting, in one octant, the inner sheet and, in the adjacent octant, the outer sheet. The continuity of both sheets through the singular point is suggested by several continuous lines on the surface, all of them passing through the singular point; but attention is not drawn to this most important feature of the surface.

Third, usually as in the second method plus a verbal description of the configuration of the surface in three dimensions; for example:

Basset (1892, p. 120): "... the wave surface ... consists of an outer and an inner sheet, which intersect at four points in the plane xz."

Courant and Hilbert (1937, p. 459): "Danach ist die Strahlenfläche eine Fläche derselben Art wie die Normalenfläche ..." and "Die Strahlenfläche zerfällt genau wie die Normalenfläche in zwei Mäntel. ..." Of the "Normalenfläche" they say (ibid., p. 458): "Der äußere Mantel ... besitzt in den vier Doppelpunkten vier nach innen gerichtete konische Ecken, während dort der innere Mantel nach außen gerichtete konische Ecken hat."

Drude (1902, p. 327): "This surface, like the normal surface, consists of two sheets. These two surfaces are very similar to each other. ..." Of the normal surface he says (ibid., p. 318): "In the xz-plane ... the two sheets of the normal surface intersect. It can be shown that *this occurs for no other directions of the wave-normal*."

Houstoun (1921, p. 208): "The wave surface ... consists of two sheets which meet in four points ... and nowhere else."

Jenkins and White (1953, p. 517): "The outer surface of the complete wave surfaces touches the inner at only four points, where it forms 'dimples'." In addition, in their sections, the inner and outer portions of the surface are divorced by the former being drawn in heavy lines.

Johannsen (1914, p. 97): "It is a warped surface ... symmetrical along the three principal axes and having four depressions lying in a single plane."

Pockels (1906, p. 44): "Die Strahlenfläche der zweiachsigen Kristalle ist eine Fläche 4. Ordnung und 4. Klasse. Sie besteht aus zwei Schalen, welche nur in den vier auf den Strahlenachsen liegenden *konischen Doppelpunkten* zusammenhängen."

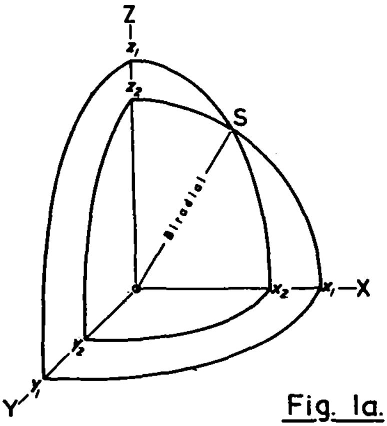
Tutton (1922, p. 866): "The wave-surface ... is a surface of the fourth degree composed of two shells, one within the other. These two shells touch each other at four points, which are at the bottom of depressions in the outer shell and on the summits of protuberances on the inner one. ... Except for these depressions and protuberances where actual contact occurs the two shells resemble ellipsoids of general triaxial form. Indeed it is common to regard the shells as two separate wave-surfaces."

Winchell (1931, p. 152): "... the complete wave-fronts are continuous surfaces intersecting in only four points ...".

Wölfling (1902, p. 364): "Die Wellenfläche besteht aus *zwei Mänteln*, welche in den Knotenpunkten aneinanderstoßen." (This paper contains a comprehensive bibliography of earlier literature on the subject.)

Wood (1934, p. 377): "The wave-surface ... consists of two sheets ..." and (ibid., p. 379) "By a little exercise of the imagination it is easy to see the general form of the inner and outer sheets. ... The outer sheet has the general form of an ellipsoid with four depressions or pits similar to the pit found on an apple around the point where the stem is inserted, only much shallower. At these four points the two sheets come in contact. ..."

Givens and Discher (1954, p. 381) carry the description a stage beyond that reached in the foregoing by providing three new sections through and near the singular points. In addition they state (ibid., p. 380): "... considerable point may be made of the fact that the outer and inner sheets of the wave surface meet at only four points"; and (ibid., p. 381), "It is not easy to imagine two sheets which can 'intersect' so cleanly . . . and yet have only point intersections, i.e., no curves of intersection. The two sheets do not intersect in the sense that two cylindrical surfaces often do, but like two wide angle cones placed one above the other and having their apexes in contact."



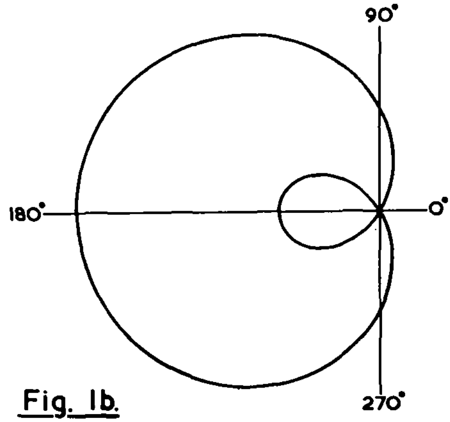


Fig. 1a. Fig. 1b.

FIG. 1*a*. Conventional three-dimensional representation of one octant of the ray surface.

FIG. 1*b*. The Limaçon of Pascal ($r = 1 - 2\cos\theta$).

Most of the descriptions in the last group have one feature in common: they tend to stress the double nature of the surface and so give the non-mathematician the impression that two surfaces are involved—two surfaces that at the four singular points intersect, touch, zusammenhängen, aneinanderstoßen, come in contact, meet, or even "intersect", according to the individual authors involved.

Rossi (1957, p. 293) gives a less misleading impression: ". . . the wave front does not consist of two separate surfaces, but of a unique continuous surface folding upon itself and intersecting in a complicated manner".

However, in none of the foregoing papers or texts is attention drawn to the essential simplicity of this extraordinary surface—a property that is to be expected of this, a very special as opposed to a general, case of the fourth degree surface:

It is one continuous surface that intersects itself at the four singular points; the points of intersection divide the surface into two unequal portions of which one is contained wholly within the other.

(The Limaçon of Pascal (fig. 1*b*), $r = b - a\cos\theta$ where $b < a$ (Walker, 1950, p. 29), is perhaps the simplest two-dimensional analogue).

This simple relationship is easy to visualize if one considers the movement of a free point upon the surface. It is obvious from the standard 3-plane representation of the surface (fig. 1*a*) that within the three coordinate planes the moving point can neither pass from the outer to the inner portion of the surface nor vice versa without passing through the singular point S.

Consider a possible path $y_1z_1Sx_2y_2z_2Sx_1y_1$ (fig. 1*a*) for the moving point. Part of this path lies on the outer, part on the inner portion of the surface; but the important point is that, in passing from the outer to the inner portion or vice versa through S, the moving point suffers no abrupt change of direction at S: there is a smooth natural passage through S along an elliptical curve (z_2Sx_1 is actually circular). Moreover, this smoothly effected passage through S from one portion of the ray surface to the other is not a peculiarity of the XZ plane: the transition is equally smoothly effected, still through S of course, regardless of the direction of transit through S and provided that there is no discontinuous change of direction at S.

To illustrate the complete continuity of the outer and inner portions of the ray surface through S a series of partial central sections (figs. 2–5) have been constructed through S in several directions (see accompanying stereogram, fig. 6). The sections have been constructed using Fletcher's (1892, p. 48) equations for the velocities, r_1 and r_2, of rays in a direction defined by its inclinations, σ_1 and σ_2, to the biradials OS_1 and OS_2 respectively:

$$\frac{1}{r_1^2} = \frac{1}{a^2}\cos^2\frac{\sigma_1-\sigma_2}{2}+\frac{1}{c^2}\sin^2\frac{\sigma_1-\sigma_2}{2} \tag{2}$$

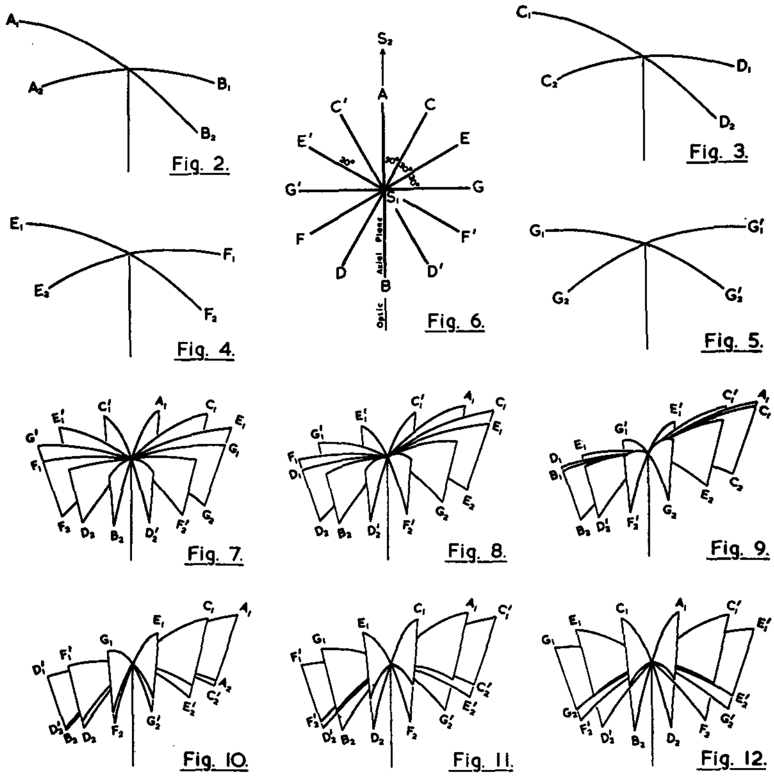
$$\frac{1}{r_2^2} = \frac{1}{a^2}\cos^2\frac{\sigma_1+\sigma_2}{2}+\frac{1}{c^2}\sin^2\frac{\sigma_1+\sigma_2}{2} \tag{3}$$

Obviously the absolute values of a, b, and c are not critical in this connexion and, merely to simplify computation, the values 2, 1·5, and 1 respectively have been used throughout. Hence, from the equation (Fletcher, 1892, p. 35)

$$\tan\tfrac{1}{2}S_1S_2 = \frac{c\sqrt{(a^2-b^2)}}{a\sqrt{(b^2-c^2)}} \tag{4}$$

the angle S_1S_2 between the two biradials, measured across Z (fig. 1), is 61° 13′.

The values of σ_1 and σ_2 for any chosen ray direction OR within any



FIGS. 2 to 5. Partial central sections of the ray surface through the biradial, OS_1 (direction represented by vertical line in each case). $A_1\ B_1\ C_1 \ldots$ and $A_2\ B_2\ C_2 \ldots$ are extremities of faster and slower rays respectively traversing directions OA, OB, OC, . . . (see fig. 6).

FIG. 6. Partial stereogram, centre S_1, showing orientation of sections in figs. 2–5 and 7–12. Section $C'D'$ is symmetrically equivalent to section CD, &c.

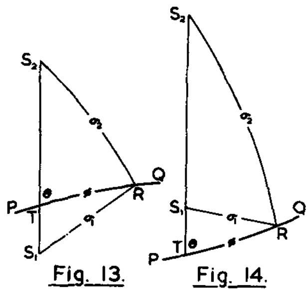
FIGS. 7 to 12. Three-dimensional representations of that part of the ray surface within 20° of the biradial, OS_1 (direction represented by vertical line in each case), prepared by orthographic projection of figs. 2, 3, 4, 5, and their symmetrical equivalents on a series of planes parallel to the biradial, OS_1. The projection lines, i.e. lines of sight, are all perpendicular to the biradial, OS_1, and lie mid-way between the following ray directions (see fig. 6): fig. 7, OB and OD'; fig. 8, OD' and OF'; fig. 9, OF' and OG; fig. 10, OG and OE; fig. 11, OE and OC; fig. 12, OC and OA.

plane of section PQ may then be calculated from the following equations (see stereograms, figs. 13 and 14):

$$\cos\sigma_1 = \cos\phi\cos S_1T \pm \sin\phi\sin S_1T\cos\theta \qquad (5)$$

$$\cos\sigma_2 = \cos\phi\cos S_2T \pm \sin\phi\sin S_2T\cos\theta \qquad (6)$$

where $\phi = TR$ = inclination of rays Or_1 and Or_2 to intersection of optic axial plane with plane of section, PQ; θ = angle S_2TR between plane of



FIGS. 13, 14. Partial stereograms, centre S_1, showing derivation of equations (5) and (6).

section PQ and optic axial plane; $S_1 S_2 = 61° 13'$; $S_2 T = 61° 13' \pm S_1 T$. For special cases, equations (5) and (6) may be modified as follows:

When $\theta = 90°$:

$$\cos\sigma_1 = \cos\phi \cos S_1 T;$$

$$\cos\sigma_2 = \cos\phi \cos S_2 T.$$

When $S_1 T = 0°$:

$$\cos\sigma_1 = \cos\phi, \text{ so } \sigma_1 = \phi;$$

$$\cos\sigma_2 = \cos\phi \cos S_1 S_2 \pm \sin\phi \sin S_1 S_2 \cos\theta.$$

When $\theta = 90°$ and $S_1 T = 0°$:

$$\cos\sigma_1 = \cos\phi, \text{ so } \sigma_1 = \phi;$$

$$\cos\sigma_2 = \cos\phi \cos S_1 S_2.$$

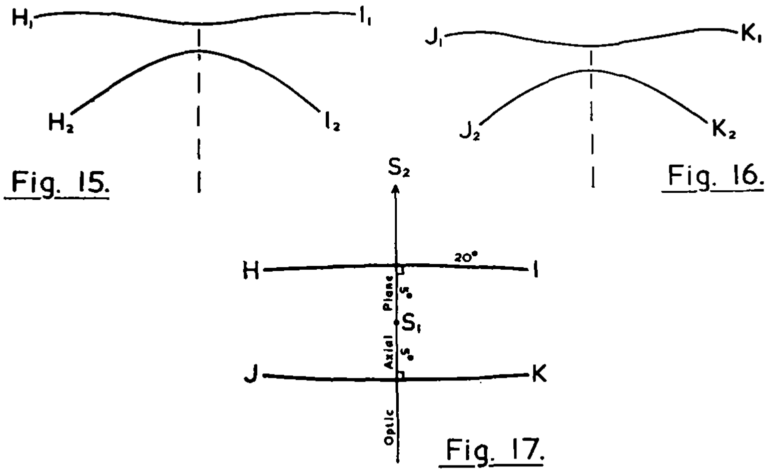
When $\theta = 0°$:

$$\cos\sigma_1 = \cos\phi, \text{ so } \sigma_1 = \phi;$$

$$\cos\sigma_2 = \cos\phi \cos S_1 S_2 \pm \sin\phi \sin S_1 S_2 = \cos(S_1 S_2 \pm \phi)$$

$$= \cos(S_1 S_2 \pm \sigma_1), \text{ so } \sigma_2 = S_1 S_2 \pm \sigma_1.$$

In addition, partial central sections (figs. 15 and 16) have been constructed perpendicular to the plane $S_1 S_2$ and crossing it 5° on either side of S_1 (see accompanying stereogram, fig. 17). These serve to illustrate that a point on the surface, although moving close to a singular point,



FIGS. 15, 16. Partial central sections of the ray surface, crossing the optic axial plane (broken vertical line) 5° each side of the biradial, OS_1 (see fig. 17).

FIG. 17. Partial stereogram, centre S_1, showing orientation of sections in figs. 15 and 16.

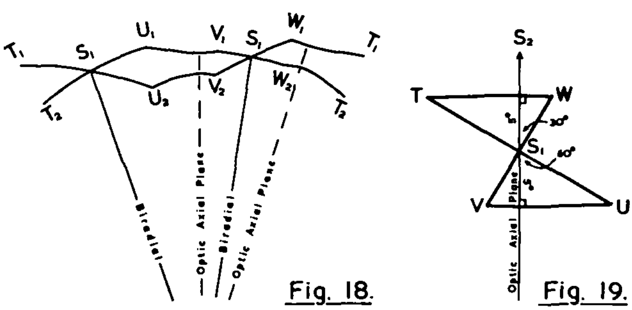


FIG. 18. Serial section of ray surface passing twice through and twice near the biradial, OS_1 (see fig. 19).

FIG. 19. Partial stereogram, centre S_1, showing orientation of serial section in fig. 18.

cannot pass from the outer to the inner portion of the surface or vice versa without passing exactly through the singular point. This is further illustrated by combining several of the foregoing sections in a serial section (fig. 18) passing twice through and twice near the point S_1 (see accompanying stereogram, fig. 19).

Finally, the sections in figs. 2–5 have, by orthographic projection on a series of planes parallel to the biradial, been combined in a series of diagrams (figs. 7–12) which show in some detail the complete continuity in three dimensions of the inner and outer portions of the ray surface through any singular point.

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