THE AMERICAN MINERALOGIST

JOURNAL OF THE MINERALOGICAL SOCIETY OF AMERICA

Vol. 26 FEBRUARY, 1941 No. 2

CRYSTALLOGRAPHIC PROCEDURES*

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During the preparation of crystallographic descriptions for the new edition of Dana's *System of Mineralogy* and also in mineral descriptions from this laboratory, certain procedures are followed. These are presented here.

Measurements

To give a concise and accurate treatment of morphological investigations a tabulation such as that in Table 1 is used. In this table "No. Times" refers to the number of times faces of the listed form were seen, and "Qual." refers to the average quality of the reflecting signal. Rare or uncertain forms are so designated, since the investigating author can determine better than the reader the validity of a form. Uncertain forms are indicated by a question mark, letters being used only for established forms.

Table 1. Presentation of Morphological Data

Form	No. Xls.	No. Times	Size	Qual.	Measured Range ϕ ρ or Other Angles	Weighted Mean ϕ ρ or Other Angles	Calculated Values ϕ ρ or Other Angles
c 001	2	4	VL*	A*			
m 110	4	8	L	B			
w 011	4	4	M	C			
d 101	1	2	S	D			
p 111	4	12	VS	E			

* VL = Very Large
L = Large
M = Medium
S = Small
VS = Very Small

A = Excellent
B = Good
C = Fair
D = Poor
E = Bad

* Contribution from the Department of Mineralogy and Petrography, Harvard University, No. 234.

Symmetry

The preferred statement of the crystal class is that of Groth (modified by A. F. Rogers, textbook, 1937), accompanied by the complete Hermann-Mauguin symbols. Corresponding Schoenflies symbols and Dana (textbook, 1932) symmetry class numbers are listed in Table 2.

Table 2. Symmetry Class Notation

System	Groth (Modified)	Hermann-Mauguin	Schoenflies	Dana (1932)
	1. Pedial	1	C_1	32
Triclinic	2. Pinacoidal	$\bar{1}$	C_i	31
	3. Domatic	m	C_s	30
Monoclinic	4. Sphenoidal	2	C_2	29
	5. Prismatic	$\frac{2}{m}$	C_{2h}	28
	6. Rhombic-pyramidal	$m\ m\ 2$	C_{2v}	26
Orthorhombic	7. Rhombic-disphenoidal	$2\ 2\ 2$	D_2	27
	8. Rhombic-dipyramidal	$\frac{2}{m}\ \frac{2}{m}\ \frac{2}{m}$	D_{2h}	25
	9. Tetragonal-disphenoidal	$\bar{4}$	S_4	12
	10. Tetragonal-pyramidal	4	C_4	9
	11. Tetragonal-dipyramidal	$\frac{4}{m}$	C_{4h}	8
Tetragonal	12. Tetragonal-scalenohedral	$\bar{4}\ 2\ m$	D_{2d}	10
	13. Ditetragonal-pyramidal	$4\ m\ m$	C_{4v}	7
	14. Tetragonal-trapezohedral	$4\ 2\ 2$	D_4	11
	15. Ditetragonal-dipyramidal	$\frac{4}{m}\ \frac{2}{m}\ \frac{2}{m}$	D_{4h}	6
	16. Trigonal-pyramidal	3	C_3	24
Hexagonal	17. Rhombohedral	$\bar{3}$	C_{3i}	22
P or *R**	18. Ditrigonal-pyramidal	$3\ m$	C_{3v}	21
	19. Trigonal-trapezohedral	$3\ 2$	D_3	23
	20. Hexagonal-scalenohedral	$\bar{3}\ \frac{2}{m}$	D_{3d}	20

* In the hexagonal system the lattice mode in classes 16–20 may be either primitive hexagonal (*P*) or rhombohedral (*R*); in classes 21–27 only the primitive mode is possible.

TABLE 2. *Continued*

System	Groth (Modified)	Hermann-Mauguin	Schoenflies	Dana (1932)
Hexagonal P^*	21. Trigonal-dipyramidal	$\bar{6}$	C_{3h}	19
	22. Hexagonal-pyramidal	6	C_6	16
	23. Hexagonal-dipyramidal	$\frac{6}{m}$	C_{6h}	15
	24. Ditrigonal-dipyramidal	$\bar{6}\ m\ 2$	D_{3h}	18
	25. Dihexagonal-pyramidal	$6\ m\ m$	C_{6v}	14
	26. Hexagonal trapezohedral	$6\ 2\ 2$	D_6	17
	27. Dihexagonal-dipyramidal	$\frac{6}{m}\ \frac{2}{m}\ \frac{2}{m}$	D_{6h}	13
Isometric	28. Tetartoidal	$2\ 3$	T	5
	29. Diploidal	$\frac{2}{m}\ \bar{3}$	T_h	2
	30. Hextetrahedral	$\bar{4}\ 3\ m$	T_d	3
	31. Gyroidal	$4\ 3\ 2$	O	4
	32. Hexoctahedral	$\frac{4}{m}\ \bar{3}\ \frac{2}{m}$	O_h	1

ELEMENTS

The elements, in general, include the direct (linear) and reciprocal (polar) axial ratios and interaxial angles plus the gnomonic projection constants (when these differ from the polar values). The elements are given relative to a chosen orientation of axes and unit parametral plane. *Orientation* refers here to the relative position of the axes of a unique elementary parallelopiped. The unit morphological cell is generally chosen to conform to the structural cell when this is known. Otherwise, the unit is chosen after a consideration of various rules such as Ungemach's rule of *total of indices*, the *law of Bravais*, the generalized law of Bravais proposed by Donnay and Harker, or, preferably, Donnay's morphological method of analysis used in conjunction with Peacock's *harmonic-arithmetic rule*. A consideration of these rules usually leads to a morphological unit which conforms with that derived by x-ray study. Exceptional treatments, however, are sometimes desirable when interrelation between species can be made clearer by an unconventional treatment. Rarely, the x-ray unit involves a complication of the morphological discussion. In such cases it is not retained.

Tabulation of Forms and Angles

The table of forms and angles following the elements should include all of the common and less common forms, with letters and important angles for each. Four categories of forms are recognized in the new edition of the Dana *System;* they are: common, less common, rare, and uncertain. These are qualitative terms referring to the occurrence in each mineral species.

Few general rules for the order of form listing can be given for all crystal systems, since each possesses restrictive peculiarities. The general order is: first, forms cutting but one crystal axis; second, those cutting two axes; third, those cutting three axes. In the hexagonal system, with four axes, a similar order is employed.

Tabulations will be found in the following pages for each crystal system, giving the order in which all possible forms in each symmetry class are listed. Complementary (or correlative) forms are grouped with their holohedral equivalents and are listed in the same order as though holohedral. In case the vertical axis is polar, the headings "lower" and "upper" are used beneath the Hermann-Mauguin symmetry class symbol. Only "upper" form indices are given, and an x in the "lower" column indicates that the form may occur as a "lower" one. In specific tabulations of the forms of such crystals the same letter is given for the "lower" as for the "upper" form, but a minus sign is placed over it ($\bar{c}$ for example). If the form is observed only as a "lower" one, its letter appears only in that column, but the indices are always those of the upper merohedral equivalent. Plus or minus signs before a form letter indicate the sign of phi of the form. Prime marks (′) to the right or left of a form letter indicate whether the form occurs to the right or left of some defined meridian. The general order for plus and minus, right and left forms is: plus right (+′), plus left (′+), minus left (′−), minus right (−′).

The angles given in the table for the representative face of each form include the azimuth phi (ϕ) and polar distance rho (ρ), angular coordinates, as well as interfacial angles to two or more fundamental faces. The angles given and their meanings will be shown in figures included in the discussion of each system.

Transformation Formulae

In the transformation of elements and indices from one set to another (old to new) the transformation matrix is written in the linear form (Barker, 1930): $u_1v_1w_1/u_2v_2w_2/u_3v_3w_3$, where $(u_1u_2u_3)$,* $(v_1v_2v_3)$, and

* The usage of brackets followed here is:
( ) = face symbol
{ } = form symbol
[ ] = zone or axis symbol

$(w_1w_2w_3)$ are the indices, multiple if need be, in the new orientation of the old axial planes (100), (010), and (001), respectively; and where $[u_1v_1w_1]$, $[u_2v_2w_2]$, and $[u_3v_3w_3]$ in the old orientation represent the new axial lengths [100], [010], and [001], respectively.

To obtain such a formula, then, it is necessary to determine the new indices of the old axial planes (100), (010), and (001) and to write them in that order in vertical columns as shown below. It is further necessary to determine the new indices of any general face (hkl) or of a pair of special faces of $(hk0)$, $(0kl)$, and $(h0l)$. This requirement arises from the fact that it is impossible to determine from a projection whether the new indices of the old axial planes are correct or are but the simplest expression of the actual multiple indices of the plane, as is demonstrated later. Any face symbol (hkl) in the old orientation may be transformed to the new equivalent $(h'k'l')$ by the formula $u_1v_1w_1/u_2v_2w_2/u_3v_3w_3$ as follows:

$$h' = u_1h + v_1k + w_1l;\ k' = u_2h + v_2k + w_2l;\ l' = u_3h + v_3k + w_3l.$$

The following example will serve to indicate the procedure followed to obtain a transformation formula.

Old Orientation	New Orientation	
(100) =	$(\bar{1}01) = (u_1u_2u_3)$	(1)
(010) =	$(0\bar{1}0) = (v_1v_2v_3)$	
(001) =	$(011) = (w_1w_2w_3)$	
(121) =	$(\bar{1}03)$	

Writing the given relations in the linear form $u_1v_1w_1/u_2v_2w_2/u_3v_3w_3$, we obtain $\bar{1}00/0\bar{1}1/101$. This formula must be checked by transforming (121) to obtain the new symbol. This is done graphically below:

$$\begin{array}{ccccccccccc}
1 & 2 & 1 & & 1 & 2 & 1 & & 1 & 2 & 1 \\
\times & \times & \times & & \times & \times & \times & & \times & \times & \times \\
\bar{1} & 0 & 0 & / & 0 & \bar{1} & 1 & / & 1 & 0 & 1 \\
\hline
 & \bar{1}+0+0 & & & & 0+\bar{2}+1 & & & & 1+0+1 & \\
 & =\bar{1} & & & & =\bar{1} & & & & =2 &
\end{array} \quad \text{(2)}$$

The new indices obtained are $(\bar{1}\bar{1}2)$ instead of the correct $(\bar{1}03)$, showing the need of adjusting the terms of the formula. By a simple cut-and-try process various multiple indices are substituted in the second column of (1). When (011) of (1) is changed to (022), the other equivalences remaining unchanged, the correct formula results.

Old	New		
(100) =	$(\bar{1}01)$		
(010) =	$(0\bar{1}0)$	or $\bar{1}00/0\bar{1}2/102$	(3)
(001) =	(022)		

Transforming (121) as in (2) we now obtain $(\bar{1}03)$ which validates formula (3).

As was indicated earlier, this type of formula not only facilitates the transformation of indices, but it also permits the determination of the new axial directions, lengths, and angles from their identity in the old lattice. $[u_1v_1w_1]$ $[u_2v_2w_2]$, and $[u_3v_3w_3]$ in the old lattice orientation are the new [100], [010], and [001], respectively. Thus, in the transformation example just given we see that:†

Old Orientation	New Orientation ....(4)
$[u_1v_1w_1]=[\bar{1}00]$	$\equiv[100]$
$[u_2v_2w_2]=[0\bar{1}2]$	$\equiv[010]$
$[u_3v_3w_3]=[102]$	$\equiv[001]$

Conventionally, the new interaxial angles are defined as follows:

New	Old
$\alpha'=[010]\wedge[001]$	$\equiv[0\bar{1}2]\wedge[102]$
$\beta'=[001]\wedge[100]$	$\equiv[102]\wedge[\bar{1}00]$
$\gamma'=[100]\wedge[010]$	$\equiv[\bar{1}00]\wedge[0\bar{1}2]$

The length, T, of any $[uvw]$, in this case of $[\bar{1}00]$, $[0\bar{1}2]$, or $[102]$ in the old lattice, is calculated as follows:

$$T^2_{uvw}=a^2u^2+b^2v^2+c^2w^2+2bc\cos\alpha+2ca\cos\beta+2ab\cos\gamma \quad \text{.............(5)}$$

($a, b, c, \alpha, \beta, \gamma$ are the linear elements of the old lattice). If τ is the angle between the directions $[u_1v_1w_1]$ and $[u_2v_2w_2]$ in the old lattice, the new axial angles are computed by the general formula:

$$\cos\tau=[a^2u_1u_2+b^2v_1v_2+c^2w_1w_2+bc(v_1w_2+w_1v_2)\cos\alpha$$
$$+ca(w_1u_2+u_1w_2)\cos\beta+ab(u_1v_2+v_1u_2)\cos\gamma]/T_{u_1v_1w_1}\cdot T_{u_2v_2w_2} \quad \text{.................(6)}$$

For a more extended discussion of transformations see the following works:

Lewis, W. J., 1899—Crystallography, 104.
Barker, T. V., 1930—Systematic Crystallography, 32–34.
Donnay, J. D. H., 1937—*Am. Mineral.*, **22,** 621–624.
Peacock, M. A., 1937—*Am. Mineral.*, **22,** 588–620.
Richmond, W. E., 1937—*Am. Mineral.*, **22,** 630–642.
Wolfe, C. W., 1937—*Am. Mineral.*, **22,** 736–741.
International Tables, 1935—**1,** 73–76.

The Triclinic System

Elements. The elements in the triclinic system include: the linear axial ratio $a:b:c$ $(b=1)$ and interaxial angles α, β, γ; the polar axial ratio $p_0:q_0:r_0$ $(r_0=1)$ and interaxial angles λ, μ, ν; and the gnomonic projection constants p_0', q_0', x_0', y_0'. In the triclinic system the gnomonic projection values are greater than the polar values, since r_0 of the polar ratio, which is taken as unity and normal to c (001), is inclined to the center of the projection.

† The sign $\equiv$ is read "equals after transformation."

The plane of projection of the polar elements thus drops an amount which is a function of the rho value of *c* (001). The gnomonic projection plane $A'B'C'D'$ (Fig. 1) drops to the position $ABCD$ of the polar elements, both parallel to the equatorial plane. The position of $ABCD$ is determined by the point of emergence on the sphere of projection of the normal to *c* (001).

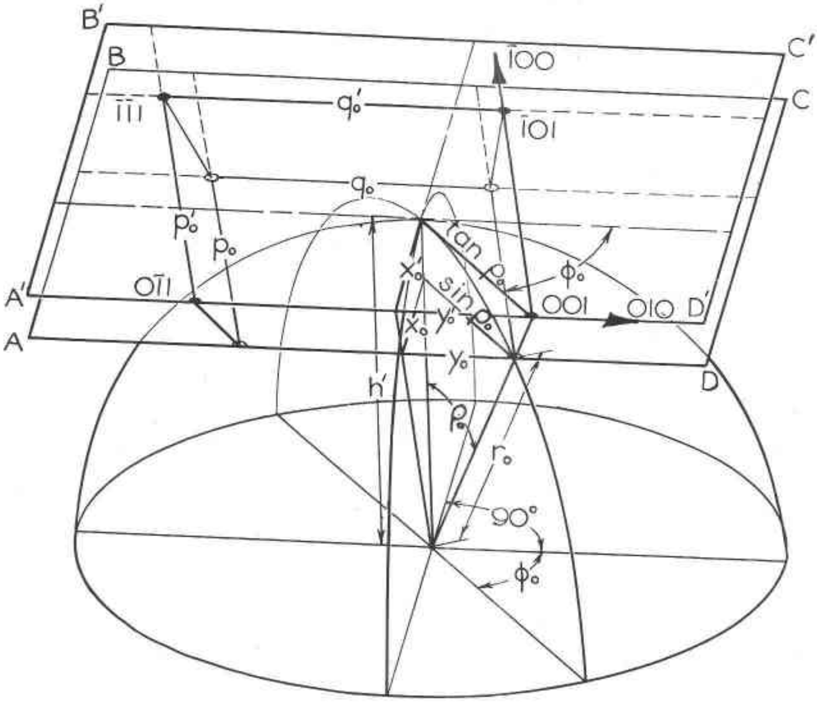

FIG. 1. Relation of Gnomonic Projection Elements to Polar Elements in the Triclinic System.

The linear interaxial angles α, β, γ are defined as follows:

$$\alpha=[010]\wedge[001],\ \beta=[001]\wedge[100],\ \gamma=[100]\wedge[010] \quad \ldots\ldots\ldots\ldots (7)$$

The polar axial angles λ, μ, ν are the following interfacial angles:

$$\lambda=(010)\wedge(001),\ \mu=(001)\wedge(100),\ \nu=(100)\wedge(010) \quad \ldots\ldots\ldots\ldots (8)$$

The six angles are shown in stereographic projection (Fig. 2).

The projection constants used in calculations are shown in Figs. 4*a* and 4*b*, which differ only in the obliquity of ν.

Calculation of V_0. The first step in the determination of the elements from two-circle goniometrical measurements is the calculation of V_0, the best average vertical circle goniometer reading for the normal to (010), which is chosen as the zero meridian. When b (010) is well developed and reflects well, its v reading may be made V_0. The best value for

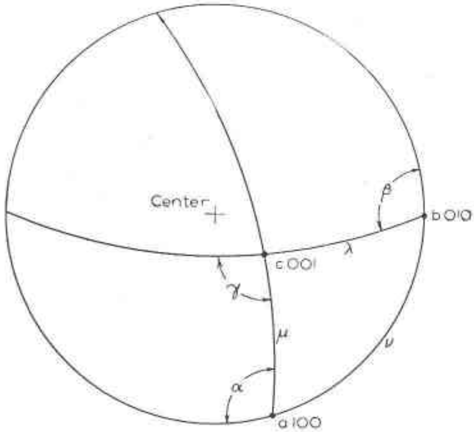

FIG. 2. Stereographic Projection of Linear and Polar Interaxial Angles in the Triclinic System.

V_0 is subtracted from the vertical circle reading of each face to obtain its ϕ. If the resulting angle be greater than 180°, it is subtracted from 360°, and the remainder constitutes a negative ϕ.

V_0 may be obtained from the v readings of three or more $(hk0)$ faces according to the formula:

$$\cot (V_0 - v_2) = Q\cot(v_3 - v_1) - (1\text{-}Q)\cot(v_2 - v_1) \ldots\ldots\ldots\ldots (9)$$

where: v_1 = vertical circle reading of (h_1k_10);
v_2 = vertical circle reading of (h_2k_20);
v_3 = vertical circle reading of (h_3k_30);

$$Q = \frac{\dfrac{k_3}{h_3} - \dfrac{k_2}{h_2}}{\dfrac{k_3}{h_3} - \dfrac{k_1}{h_1}}$$

V_0 is also calculated from measurements of various pairs of terminal faces, each pair having the same h/l value, according to the following formula:

$$\text{Tan } V_0 = \frac{(\sin v_2 \tan \rho_2) - (\sin v_1 \tan \rho_1)}{(\cos v_2 \tan \rho_2) - (\cos v_1 \tan \rho_1)} \ldots\ldots\ldots\ldots (10)$$

where v_1, ρ_1; v_2, ρ_2 are the angular readings of the two faces (Fig. 3). This calculation is repeated for several pairs of such faces, and the results averaged. Due attention must be given to the signs of the trigonometric functions.

Extended demonstrations and illustrations of the use of (9) and (10) are given by C. Dreyer and V. Goldschmidt—*Meddelelser om Grønland*,

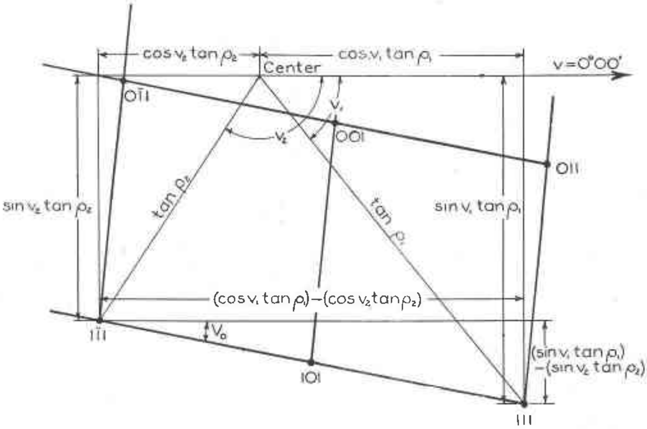

FIG. 3. Gnomonic Projection to Illustrate Calculation of V_0 from the Angular Readings of Two Terminal Forms with the same h/l value.

34, 29–36 (1907); by L. Borgstrom and V. Goldschmidt—*Zeits. Kryst.*, **41,** 75–78 (1905); and by A. L. Parsons—*Am. Mineral.*, **5,** 193-194, 198–203 (1920).

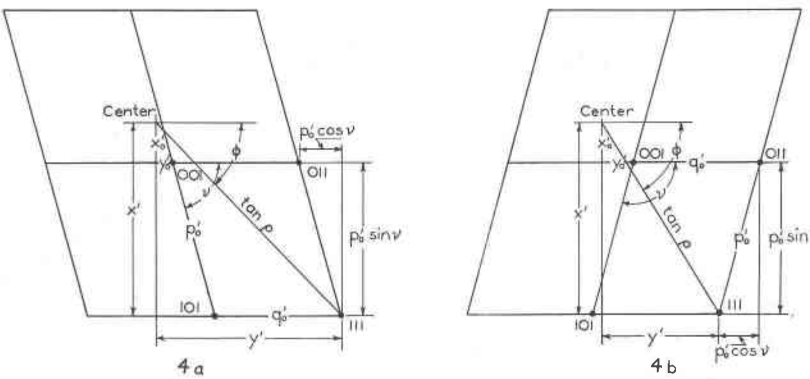

FIG. 4. Gnomonic Projection Constants in the Triclinic System with ν less than 90°00′ (4a) and greater than 90°00′ (4b).

Calculation of Projection Elements. The following formulae are useful in the calculation of the projection elements p_0', q_0', x_0', y_0', from two-circle goniometric measurements (see Fig. 4).

$$x' = \sin\phi \tan\rho = x_0' + \frac{h}{l} p_0' \sin\nu \text{ from Fig. 4} \quad \ldots\ldots (11)$$

$$y' = \cos\phi \tan\rho = y_0' + \frac{k}{l} q_0' + \frac{h}{l} p_0' \cos\nu \text{ from Fig. 4} \quad \ldots\ldots (12)$$

If x_1', x_2', . . . x_n' are the ordinates and y_1', y_2', . . . y_n' the abscissae of the gnomonic poles $(h_1k_1l_1)$, $(h_2k_2l_2)$, . . . $(h_nk_nl_n)$, then:

$$p_0' \sin\nu = \frac{l_1l_2(x_1' - x_2')}{h_1l_2 - h_2l_1} \text{ from (11)} \quad \ldots\ldots (13)$$

$$x_0' = x' - \frac{h}{l} p_0' \sin\nu \text{ from (11)} \quad \ldots\ldots (14)$$

$$p_0' \cos\nu = \frac{l_1l_2(y_1' - y_2')(k_2l_3 - k_3l_2) - l_2l_3(y_2' - y_3')k_1l_2 - k_2l_1}{(h_1l_2 - h_2l_1)(k_2l_3 - k_3l_2) - (h_2l_3 - h_3l_2)(k_1l_2 - k_2l_1)} \quad \ldots\ldots (15)$$

$$q_0' = \frac{p_0' \cos\nu(h_2l_1 - h_1l_2) + l_1l_2(y_1' - y_2')}{k_1l_2 - k_2l_1} \text{ from (12)} \quad \ldots\ldots (16)$$

$$y_0' = y' - \frac{k}{l} q_0' - \frac{h}{l} p_0' \cos\nu \text{ from (12)} \quad \ldots\ldots (17)$$

$$\tan\nu = \frac{p_0' \sin\nu}{p_0' \cos\nu} \quad \ldots\ldots (18)$$

$$p_0' = \frac{p_0' \sin\nu}{\sin\nu} = \frac{p_0' \cos\nu}{\cos\nu} \quad \ldots\ldots (19)$$

Relation of Projection Elements to Polar Elements. ϕ_0 and ρ_0 are the azimuth and the polar distance of $c\,(001)$, the rectangular coordinates of which are x_0' and y_0' (Fig. 5).

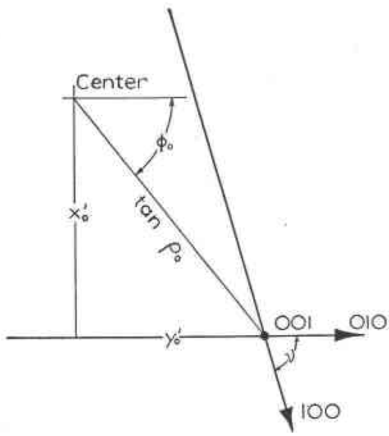

FIG. 5. Angular and Rectangular Coordinates of c (001) in the Triclinic System.

$$\tan\phi_0 = \frac{x_0'}{y_0'} \text{ from Fig. 5} \quad \ldots\ldots (20)$$

$$= \frac{x_0}{y_0} \text{ from Figs. 1 and 5}$$

(projection values are primed, and polar values are unprimed)

$$\tan \rho_0 = \frac{x_0'}{\sin \phi_0} = \frac{y_0'}{\cos \phi_0} \text{ from Fig. 5} \quad (21)$$

$$\sin \rho_0 = \frac{x_0}{\sin \phi_0} = \frac{y_0}{\cos \phi_0} \text{ from Fig. 1} \quad (22)$$

$$x_0 = \sin \rho_0 \sin \phi_0 \quad (23)$$

$$y_0 = \sin \rho_0 \cos \phi_0 \quad (24)$$

$$p_0 = p_0' \cos \rho_0 \text{ from Fig. 1} \quad (25)$$

$$q_0 = q_0' \cos \rho_0 \text{ from Fig. 1} \quad (26)$$

$$\cos \lambda = y_0 \text{ from Fig. 2} \quad (27)$$

$$\cos \mu = y_0 \cos \nu + x_0 \sin \nu \text{ from Fig. 2} \quad (28)$$

The linear and polar elements are related as follows:

$$a:b(=1):c::\frac{\sin \alpha}{p_0}:\frac{\sin \beta}{q_0}:\frac{\sin \gamma}{r_0(=1)}::\frac{\sin \lambda}{p_0}:\frac{\sin \mu}{q_0}:\frac{\sin \nu}{r_0(=1)} \quad (29)$$

$$a = \frac{q_0' \sin \lambda}{p_0' \sin \mu} \text{ from (25), (26), (29)} \quad (30)$$

$$c = \frac{q_0' \cos \nu \sin \lambda}{\sin \mu} \text{ from (26) and (29)} \quad (31)$$

$$p_0 = \frac{c \sin \alpha}{a \sin \gamma} \text{ from (29)} \quad (32)$$

$$q_0 = \frac{c \sin \beta}{\sin \gamma} \text{ from (29)} \quad (33)$$

The relationship between the polar angles λ, μ, ν and the linear angles α, β, γ is as follows

$$\left(\sigma = \frac{\lambda+\mu+\nu}{2}\right):$$

$$\sin \frac{\alpha}{2} = \sqrt{\frac{\sin \sigma \sin (\sigma - \lambda)}{\sin \mu \sin \nu}} \text{ from Fig. 2} \quad (34)$$

$$\sin \frac{\beta}{2} = \sqrt{\frac{\sin \sigma \sin (\sigma - \mu)}{\sin \nu \sin \lambda}}$$

$$\sin \frac{\gamma}{2} = \sqrt{\frac{\sin \sigma \sin (\sigma - \nu)}{\sin \lambda \sin \mu}}$$

To obtain λ, μ, ν from α, β, γ, substitute the latter for the former, respectively, in (34).

Order of Form Listing. The following order of listing (Table 3) is used in the triclinic system. The letters c, b, a, m, M are reserved for the faces (001), (010), (100), (110), $1\bar{1}0$), respectively.

TABLE 3. ORDER OF FORM LISTING IN THE TRICLINIC SYSTEM

Class	1		$\bar{1}$
Lower	Upper		
$\bar{c}$	c	001	001
	b	010	010
	$-b$	$0\bar{1}0$	
	a	100	100
	$-a$	$\bar{1}00$	
	d	$hk0$	$hk0$ in order of increasing $\frac{h}{k}$
	$-d$	$\bar{h}\bar{k}0$	
	D	$h\bar{k}0$	$h\bar{k}0$ in order of decreasing $\frac{h}{k}$
	$-D$	$\bar{h}k0$	
$\bar{w}$	w	$0kl$	$0kl$ in order of increasing $\frac{k}{l}$
$\bar{W}$	W	$0\bar{k}l$	$0\bar{k}l$ in order of increasing $\frac{k}{l}$
$\bar{e}$	e	$h0l$	$h0l$ in order of increasing $\frac{h}{l}$
$\bar{E}$	E	$\bar{h}0l$	$\bar{h}0l$ in order of increasing $\frac{h}{l}$
x	x*	hhl	hhl in order of increasing $\frac{h}{l}$
x	x	$h\bar{h}l$	$h\bar{h}l$ in order of increasing $\frac{h}{l}$
x	x	$\bar{h}hl$	$\bar{h}hl$ in order of increasing $\frac{h}{l}$
x	x	$\bar{h}\bar{h}l$	$\bar{h}\bar{h}l$ in order of increasing $\frac{h}{l}$
x	x	hkl	hkl in order of increasing $\frac{h}{l}$
x	x	$h\bar{k}l$	$h\bar{k}l$ in groups of equal $\frac{h}{l}$, list
x	x	$\bar{h}kl$	$\bar{h}kl$ in order of increasing $\frac{k}{l}$
x	x	$\bar{h}\bar{k}l$	$\bar{h}\bar{k}l$

* x indicates possible occurrence of form.

Triclinic Angles. The angles given in triclinic tables are: ϕ, ρ, A, B, C. Figure 6 is a stereographic projection showing these angles.

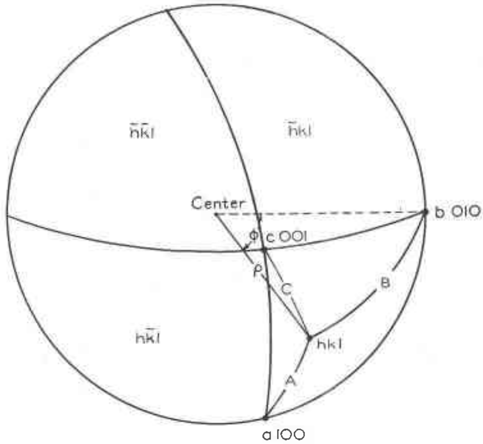

FIG. 6. Stereographic Projection of Angles Given for Forms in the Triclinic System.

$$\tan \phi_{hkl} = \frac{x'}{y'} = \frac{x_0' + \frac{h}{l} p_0' \sin \nu}{y_0' + \frac{k}{l} q_0' + \frac{h}{l} p_0' \cos \nu} \text{ from Fig. 4} \ldots\ldots\ldots\ldots (35)$$

$$\tan \rho_{hkl} = \frac{x'}{\sin \phi} = \frac{y'}{\cos \phi} \text{ from Fig. 4 and (35)} \ldots\ldots\ldots\ldots (36)$$

The angles A, B, C, are the angles which the face makes with a (100), b (010), c (001), respectively. These angles are calculated by means of the general formula for the interfacial angle Δ:

$$\cos \Delta = \cos \rho_1 \cos \rho_2 + \sin \rho_1 \sin \rho_2 \cos (\phi_2 - \phi_1) \ldots\ldots\ldots\ldots (37)$$

where ϕ_1, ρ_1 and ϕ_2, ρ_2 are the angular coordinates of the two faces. For the specific cases of the calculations of the angles A, B, C, formula (37) becomes:

$$\cos A = \sin \rho_{hkl} \cos (\phi_{100} - \phi_{hkl}) \ldots\ldots\ldots\ldots (38)$$

$$\cos B = \sin \rho_{hkl} \cos \phi_{hkl} \ldots\ldots\ldots\ldots (39)$$

$$\cos C = \cos \rho_{hkl} \cos \rho_0 + \sin \rho_{hkl} \sin \rho_0 \cos (\phi_0 - \phi_{hkl}) \ldots\ldots\ldots\ldots (40)$$

A graphical check is always made to avoid gross errors. The only source of error in the determination of ϕ and ρ is the calculation of the x' and y' coordinates. These may be checked satisfactorily on a gnomonic projection with a unit circle radius of 10 cm., using the projection constants for plotting the faces. The interfacial angles A, B, C, are checked on a stereographic net (20 cm. radius), using the calculated ϕ and ρ values for plotting the faces. This check is accurate to $\pm$ 15 minutes.

THE MONOCLINIC SYSTEM

Elements. The elements here consist of: the linear axial ratio, $a:b:c$ ($b=1$), and the axial angle β between the positive ends of the c and a

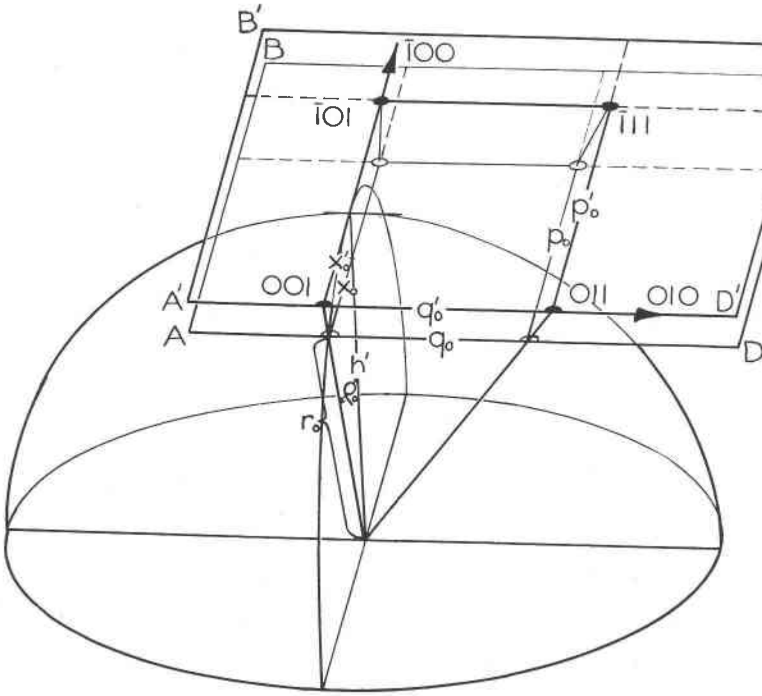

FIG. 7. Relation of Gnomonic Projection Elements to Polar Elements in the Monoclinic System.

axes; the polar ratio, $p_0:q_0:r_0$ ($r_0=1$), and the reciprocal axial angle μ (the supplement of β); the polar ratio $r_2:p_2:q_2$ ($q_2=1$), obtained when b (010) is set in polar position, and the phi of a (100) is $0°00'$; the projection constants p_0', q_0', x_0' (see Fig. 8).

In the monoclinic system, as in the triclinic, the projection elements do not coincide with the polar elements. For $r_0=1$ the gnomonic plane $A'B'C'D'$ (Fig. 7) must drop to the position of $ABCD$, the amount of that drop being a function of the ρ angle of c (001), (ρ_0).

Once the poles of measured faces have been plotted on the gnomonic projection and the correct gnomonic net drawn, the following convention is observed: $c<a$ (b is fixed by symmetry). If the lattice is primitive, the base is the node of the gnomonic net closest to the center of projection;

this yields the least anorthism of the axes. If the lattice is centered, the base is usually chosen in such a manner that the lattice is base-centered, whether or not this makes $c<a$.

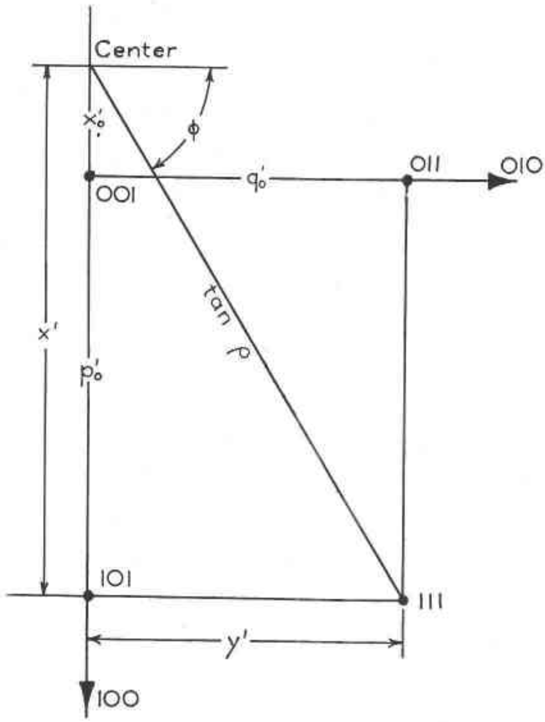

FIG. 8. Gnomonic Projection Constants in the Monoclinic System, Conventional Orientation.

Calculation of Projection Elements. The ϕ and ρ angles of the various faces are obtained from measurements. From Fig. 8 are derived the following:

$$x' = x_0' + \frac{h}{l} p_0' = \sin\phi \tan\rho \text{ (Fig. 8)} \qquad (41)$$

If x_1' and x_2' are the x' coordinates of faces $(h_1k_1l_1)$ and $(h_2k_2l_2)$, then:

$$p_0' = \frac{l_1l_2(x_1' - x_2')}{(h_1l_2 - h_2l_1)} \text{ (from 41)} \qquad (42)$$

$$x_0' = x' - \frac{h}{l} p_0' \text{ (from 41)} \qquad (43)$$

$$y' = \frac{k}{l} q_0' = \cos\phi \tan\rho \text{ (Fig. 8)} \qquad (44)$$

$$q_0' = \frac{ly'}{k} \text{ (from 44)} \quad (45)$$

$$\cot \mu = x_0' \quad (46)$$

$$\tan \rho_0 = x_0' \quad (47)$$

Calculation of Polar Elements. The polar elements are related to the projection elements as follows:

$$p_0 = p_0' \cos \rho_0 = p_0' \sin \mu \text{ (Fig. 7)} \quad (48)$$

$$q_0 = q_0' \cos \rho_0 = q_0' \sin \mu \text{ (Fig. 7)} \quad (49)$$

$$r_0 = 1 \quad (50)$$

$$r_2 = \frac{1}{q_0} = \frac{1}{q_0' \sin \mu} \quad (51)$$

$$p_2 = \frac{p_0}{q_0} = \frac{p_0'}{q_0'} \quad (52)$$

$$q_2 = 1 \quad (53)$$

Calculation of Linear Elements. The linear elements are related as follows to the polar and projection elements:

$$a = \frac{q_0}{p_0 \sin \mu} = \frac{q_0}{p_0 \ \sin \mu} \text{ (from 29)} \quad (54)$$

$$c = \frac{q_0}{\sin \mu} = q_0' \text{ (from 29)} \quad (55)$$

$$\beta = 180° - \mu \quad (56)$$

The derivation of the elements of a monoclinic mineral from measurements made with the *b*-axis vertical is as follows.

V_0 may be obtained as in (9) and (10). If, however, the quality of either *c* (001) or *a* (100) is sufficiently good, the vertical circle reading of one of them may be taken as V_0. The ϕ_2 readings for all of the faces are obtained by subtracting V_0 from their vertical circle readings (see Peacock—*Am. Jour. Sci.*, **28,** 241–254, for meaning of subscript $_2$). Ordinarily, the elements are calculated with the azimuth ϕ_2 of (100) = 0°00′ (Fig. 9), and the angles are given accordingly; but they may be calculated as readily when the azimuth of (001) = 0°00′. Two sets of similar formulae are given below for these two cases. Using the measured ϕ_2 and ρ_2 values, the x and y coordinates of each face in this orientation are obtained by (11) and (12). In the formulae below x_1 and x_2 are the ordinates and y_1 y_2 are the abscissae of any two gnomonic poles $(h_1k_1l_1)$ $(h_2k_2l_2)$. Indices obtained from the gnomonic plot in this orientation are listed in the sequence of the intercepts of the gnomonic pole on the polar axes p_2, $q_2 = 1$, and r_2, clearing fractions where necessary. Thus, these indices are identical with those obtained for the same face in the normal position where $r_0 = 1$.

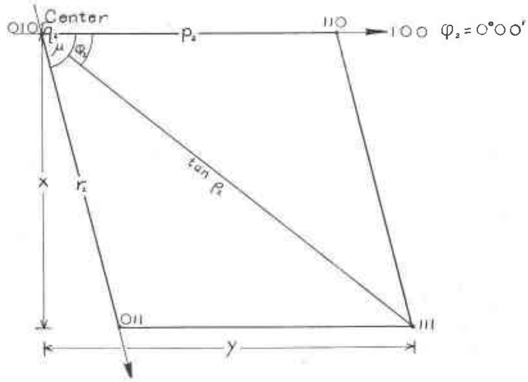

FIG. 9. Gnomonic Projection Constants in the Monoclinic System with b-axis vertical.

where ϕ_2 of a (100) = 0°00′

$$\tan\mu=\frac{k_1h_2x_1-k_2h_1x_2}{k_1h_2y_1-k_2h_1y_2}\quad\ldots(57)$$

$$r_2=\frac{kx}{l\sin\mu}\quad\ldots\ldots\ldots\ldots(58)$$

$$p_2=\frac{ky}{h}-\frac{kx}{h\tan\mu}\quad\ldots\ldots(59)$$

where ϕ_2 of c (001) = 0°00′

$$\tan\mu=\frac{k_1l_2x_1-k_2l_1x_2}{k_1l_2y_1-k_2l_1y_2}\quad\ldots(60)$$

$$r_2=\frac{ky}{l}-\frac{kx}{l\tan\mu}\quad\ldots\ldots(61)$$

$$p_2=\frac{kx}{h\sin\mu}\quad\ldots\ldots\ldots\ldots(62)$$

$$a=\frac{1}{p_2\sin\mu},\quad c=\frac{1}{r_2\sin\mu}\quad\ldots\ldots(63)$$

$$p_0'=\frac{p_2}{r_2\sin\mu},\quad q_0'=\frac{1}{r_2\sin\mu},\quad x_0'=\cot\mu\quad\ldots\ldots(64)$$

Order of Form Listing. The order of listing in the monoclinic system is shown in Table 4. The letters c, b, a, m are conventionally used for (001), (010), (100), (110), respectively.

Monoclinic Angles. The angles given in the monoclinic system are ϕ, ρ, ϕ_2, $\rho_2=B$, C, A (Fig. 10). ϕ_2 and ρ_2 are the phi and rho values obtained when b(010) is set in polar position and the azimuth of a(100) is 0°00′ (in this position the $r_2\!:\!p_2\!:\!q_2$ polar ratio is obtained). C and A represent the angles which a face makes with c(001) and a(100), respectively,

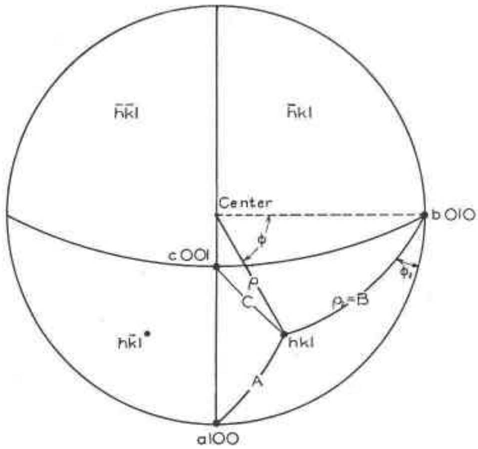

FIG. 10. Stereographic Projection of Angles Given in the Monoclinic System.

$\tan\phi=\dfrac{x'}{y'}$, (from Fig. 8) . . . . (65)

$\tan\rho=\dfrac{x'}{\sin\phi}=\dfrac{y'}{\cos\phi}$ (from Fig. 8) . . . . (66)

$\cot\phi_2=x'=\sin\phi\tan\rho$ from (41), Fig. 9, Fig. 10 . . . . (67)

$\cos\rho_2=\cos B=\sin\rho\cos\phi$ from Fig. 9, Fig. 10 . . . . (68)

$\cos C=\sin(\phi_2+\rho_0)\sin B$ from Fig. 10 . . . . (69)

$\cos A=\sin\rho\sin\phi$ from Fig. 10 . . . . (70)

TABLE 4. ORDER OF FORM LISTING IN THE MONOCLINIC SYSTEM

Class	m		2		$\dfrac{2}{m}$
Lower	Upper				
x	c	001		001	001
	b	010		010	010
			$-b$	$0\bar{1}0$	
	a	100		100	100
	$-a$	$\bar{1}00$			
	k	$hk0$	k'	$hk0$	$hk0$ in order of increasing phi
	$-k$	$\bar{h}k0$	$'k$	$h\bar{k}0$	
x	w	$0kl$	w'	$0kl$	$0kl$ in order of increasing rho
			$'w$	$0\bar{k}l$	

TABLE 4. *Continued*

Class	m	2	$\frac{2}{m}$
Lower x	Upper d $h0l$	$h0l$	$h0l$ in order of increasing $\frac{h}{l}$
x	D $\bar{h}0l$	$\bar{h}0l$	$\bar{h}0l$ in order of increasing $\frac{h}{l}$
x	q hhl	hhl	hhl in order of increasing $\frac{h}{l}$
x	Q $\bar{h}hl$	$\bar{h}hl$	$\bar{h}hl$ in order of increasing $\frac{h}{l}$
x	r hkl	r' hkl $'r$ $h\bar{k}l$	hkl in order of increasing $\frac{h}{l}$; in groups of equal $\frac{h}{l}$, list in order of increasing $\frac{k}{l}$
x	R $\bar{h}kl$	R' $\bar{h}kl$ $'R$ $\bar{h}\bar{k}l$	$\bar{h}kl$ same as for hkl

THE ORTHORHOMBIC SYSTEM

Elements. The elements in the orthorhombic system include: the axial ratio $a:b:c$ ($b=1$) (with the convention usually adopted that $c<a<b$); the polar ratio $p_0:q_0:r_0$ ($r_0=1$); the polar ratio $q_1:r_1:p_1$ ($p_1=1$); the polar ratio $r_2:p_2:q_2$ ($q_2=1$) (See Peacock—*Am. Jour. Sci.*, **28,** 241–254 [1934]. See, also, Fig. 11). Cyclic permutations of the polar ratio are given because an orthorhombic crystal may be measured with any one of the three axes vertical, and the ensuing gnomonic plot would change accordingly.

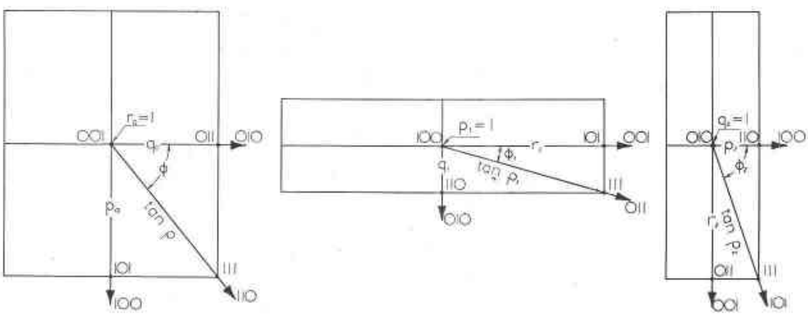

FIG. 11. Gnomonic Projections of Cyclic Permutations of Polar Elements in the Orthorhombic System (Polar Elements of Cannizzarite).

Calculation of Elements. The angular measurements, ϕ and ρ, are related to the linear and polar elements as follows:

$$a=\frac{q_0}{p_0}=\frac{k\cot\phi}{h}\ \text{(Fig. 11)} \qquad (71)$$

$$p_0=\frac{l\sin\phi\tan\rho}{h}\ \text{(Fig. 11)} \qquad (72)$$

$$q_0=c=\frac{l\cos\phi\tan\rho}{k}\ \text{(Fig. 11)} \qquad (73)$$

Order of Form Listing. The order of listing in Table 5 is followed in the orthorhombic system. The letters *c, b, a, m,* are reserved for the faces (001), (010), (100), (110), respectively. The 90° meridian defines right and left forms (Fig. 12).

Table 5. Order of Form Listing in the Orthorhombic System

Class $m\ m\ 2$		$2\ 2\ 2$	$\frac{2}{m}\ \frac{2}{m}\ \frac{2}{m}$
Lower	Upper		
x	c 001	001	001
	b 010	010	010
	a 100	100	100
	k $hk0$	$hk0$	$hk0$ in order of increasing $\frac{h}{k}$
x	w $0kl$	$0kl$	$0kl$ in order of increasing $\frac{k}{l}$
x	d $h0l$	$h0l$	$h0l$ in order of increasing $\frac{h}{l}$
x	q hhl	q' hhl $'q$ $h\bar{h}l$	hhl in order of increasing $\frac{h}{l}$
x	r hkl	r' hkl $'r$ $h\bar{k}l$	hkl in order of increasing $\frac{h}{l}$; in groups of equal $\frac{h}{l}$, list in order of increasing $\frac{k}{l}$

Orthorhombic Angles. Six angles for each form are given in the orthorhombic system. These are: ϕ and ρ in the conventional orientation ($c<a<b$); ϕ_1 and $\rho_1=A$, with $a(100)$ in polar position and the phi of $c(001)$ equal to 0°00′; and ϕ_2 and $\rho_2=B$, with $b(010)$ in polar position and

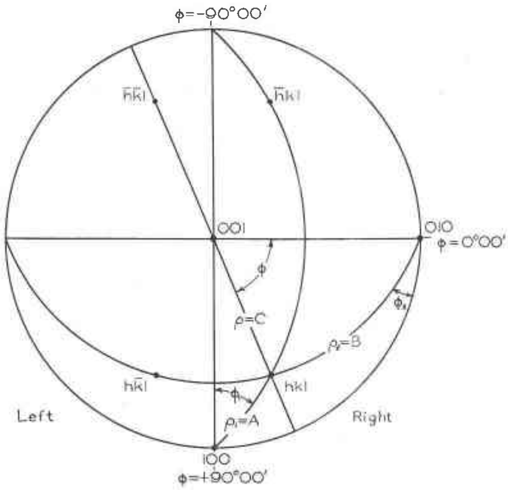

FIG. 12. Stereographic Projection Showing Form Designation and Angles in the Orthorhombic System.

the phi of $a(100)$ equal to 0°00′ (see Fig. 12). Not only do these angles give important interfacial angles, but they also provide a ready check on measured angles regardless of which axis is made vertical.

$$\tan \phi=\frac{h\,p_0}{k\,q_0} \text{ from Fig. 11} \qquad (74)$$

$$\tan \rho=C=\frac{h\,p_0}{l \sin \phi}=\frac{k\,q_0}{l \cos \phi} \text{ from Fig. 11} \qquad (75)$$

$$\tan \phi_1=\frac{k\,q_0}{l} \text{ from Fig. 12} \qquad (76)$$

$$\cos \rho_1=\cos A=\sin \rho \sin \phi \text{ from Fig. 12} \qquad (77)$$

$$\cot \phi_2=\frac{h\,p_0}{l} \text{ from Fig. 12} \qquad (78)$$

$$\cos \rho_2=\cos B=\sin \rho \cos \phi \text{ from Fig. 12} \qquad (79)$$

TETRAGONAL SYSTEM

Elements. The elements in this system include the linear ratio, $a:c$ ($a=1$), and the polar ratio $p_0:r_0$ ($r_0=1$). In this case c and p_0 are equal and may be obtained from the ϕ and ρ readings of the various types of forms as follows:

$$\text{from } (0kl) \text{ or } (k0l)\ c = p_0 = \frac{l \tan \rho}{k} \qquad (80)$$

$$\text{from } (hhl)\ c = p_0 = \frac{l \tan \rho \sin 45^\circ}{h} \qquad (81)$$

$$\text{from } (hkl)\ c = p_0 = \frac{l \tan \rho \sin \phi}{h} = \frac{l \tan \rho \cos \phi}{k} \qquad (82)$$

Order of Form Listing. Table 6 shows the order in the tetragonal system. The letters c, a, m, are reserved for the faces (001), (010), (110), respectively. The plus 45° and minus 45° meridians define right and left forms (Fig. 13).

TABLE 6. ORDER OF FORM LISTING IN THE TETRAGONAL SYSTEM

Class $\bar{4}$		4		$\frac{4}{m}$	$\bar{4}\,2\,m$	$4\,m\,m$		$4\,2\,2$	$\frac{4}{m}\,\frac{2}{m}\,\frac{2}{m}$
		Lower	Upper			Lower	Upper		
c	001	x	001	001	001	x	001	001	001
a	010		010	010	010		010	010	010
m	110		110	110	110		110	110	110
d	$hk0$		$hk0$	$hk0$	$hk0$		$hk0$	$hk0$	$hk0$ in order of increasing $\frac{h}{k}$
$-d$	$\bar{h}k0$		$\bar{h}k0$	$\bar{h}k0$					
i'	$0kl$	x	$0kl$	$0kl$	$0kl$	x	$0kl$	$0kl$	$0kl$ in order of increasing $\frac{k}{l}$
$'i$	$k0l$								
o	hhl	x	hhl	hhl	hhl	x	hhl	hhl	hhl in order of increasing $\frac{h}{l}$
$-o$	$\bar{h}hl$				$\bar{h}hl$				
u'	hkl	x	hkl	hkl	hkl	x	hkl	hkl	hkl in order of increasing $\frac{h}{l}$;
$'u$	khl	x	khl	khl				khl	in groups of equal $\frac{h}{l}$, list
$-'u$	$\bar{h}kl$				$\bar{h}kl$				in order of increasing $\frac{k}{l}$
$-u'$	$\bar{k}hl$								

Tetragonal Angles. The angles given in the tetragonal system are: ϕ; ρ; A, the angle to $a(100)$; $\overline{M}$, the angle to $-m(1\bar{1}0)$, given for plus forms only; M, the angle to $m(110)$, given for minus forms only. These

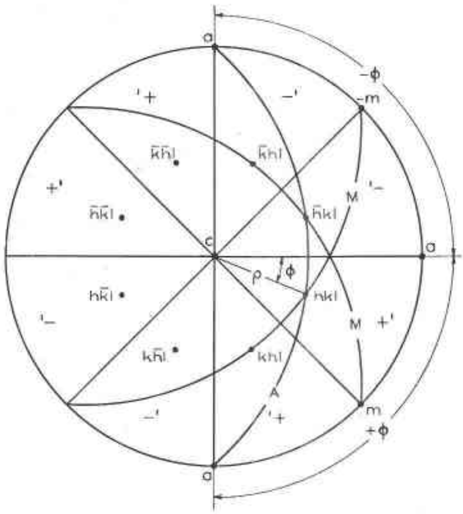

FIG. 13. Stereographic Projection Showing Form Designation and Angles in the Tetragonal System.

yield useful interfacial angles as is seen in Fig. 13. The formulae for calculation of these angles are:

$\tan\phi = \dfrac{h}{k}$ from Fig. 13 . . . . (83)

$\tan\rho = \dfrac{hp_0}{l\sin\phi} = \dfrac{kp_0}{l\cos\phi}$ from Fig. 13 . . . . (84)

$\cos A = \sin\rho\,\sin\phi$; supplement is obtained when ϕ is negative, that is,
for $\{\bar{h}kl\}$ and $\{\bar{k}hl\}$ forms . . . . (85)

$\cos\overline{M} = \sin\rho\,\cos(45°+\phi)$; supplement is obtained when form symbol is $\{khl\}$ . . . . (86)

$\cos M = \sin\rho\,\cos(45°-\phi)$; supplement is obtained when form symbol is $\{\bar{k}hl\}$ . . . . (87)

Since ϕ is a constant for each h/k value in the tetragonal system, the following table of ϕ angles is given with variations in h and k from 1 to 11. The angles are given according to increasing magnitude to facilitate rapid comparison with measured values. The same angles are valid in the isometric system.

Table 7. Phi Angles in the Tetragonal System

$h:k$	ϕ	$h:k$	ϕ	$h:k$	ϕ
1:11	5°11′40″	4:11	19°58′59″	7:10	34°59′31″
1:10	5 42 38	3: 8	20 33 22	5: 7	35 32 16
1: 9	6 20 25	2: 5	21 48 05	8:11	36 01 39
1: 8	7 07 30	3: 7	23 11 55	3: 4	36 52 12
1: 7	8 07 48	4: 9	23 57 43	7: 9	37 52 30
1: 6	9 27 44	5:11	24 26 38	4: 5	38 39 35
2:11	10 18 17	1: 2	26 33 54	9:11	39 17 22
1: 5	11 18 36	6:11	28 36 38	5: 6	39 48 20
2: 9	12 31 44	5: 9	29 03 17	6: 7	40 36 05
1: 4	14 02 10	4: 7	29 44 42	7: 8	41 11 09
3:11	15 15 18	3: 5	30 57 50	8: 9	41 37 37
2: 7	15 56 44	5: 8	32 00 19	9:10	41 59 14
3:10	16 41 57	7:11	32 28 16	10:11	42 16 25
1: 3	18 26 06	2: 3	33 41 24	1: 1	45 00 00

Hexagonal System

Introduction. The hexagonal system has caused considerable difficulty, principally due to the introduction of the so-called G_2 position of V. Goldschmidt. The G_1 position is, however, without ambiguity, as has been pointed out by Peacock (*Am. Mineral.*, **23**, 314–328, 1938) if certain changes of the polar axes and prime meridian of Goldschmidt are made. In Goldschmidt's works the G_2 position is sometimes the only one used. It is important, therefore that the transformation from the G_2 Bravais symbol to the G_1 Bravais symbol be given. It is as follows:

$$\begin{array}{cccc} h & k & \bar{\imath} & l \\ \frac{1}{3}\,\frac{2}{3}\,0\,0\,/ & \frac{1}{3}\,\frac{\bar{1}}{3}\,0\,0\,/ & \bar{\imath}\,/ & 0001 \end{array} \qquad (88)$$

where i equals $(h+k)$.

Elements. The elements of the hexagonal system include the axial ratio, $a:c$ $(a=1)$, and the polar ratio, $p_0:r_0$ $(r_0=1)$. Some of the more important mathematical relations here involved are as follows:

$$p_0 = \text{tangent } \rho\ (10\bar{1}1) \qquad (89)$$

$$c = \frac{p_0}{2}\sqrt{3} = \text{tangent } \rho\ (11\bar{2}2) \qquad (90)$$

The elements p_0 or c may be obtained from the ϕ and ρ angles of faces intersecting three axes or more, one of which must be the c-axis, by the following formulae:

$$\text{from } (h0\bar{h}l) \text{ or its equivalents, } p_0=\frac{l\tan\rho}{h};\ c=\frac{l\sqrt{3}\tan\rho}{2h} \quad\ldots\ldots\ldots\ldots(91)$$

$$\text{from } (hh\,\overline{2h}\,l) \text{ or equivalents, } p_0=\frac{l\tan\rho}{h\sqrt{3}};\ c=\frac{l\tan\rho}{2h} \quad\ldots\ldots\ldots\ldots(92)$$

$$\text{from } (hk\bar{i}l) \text{ or equivalents, } p_0=\frac{l\tan\rho}{\sqrt{h^2+k^2+hk}};\ c=\frac{l\sqrt{3}\tan\rho}{2\sqrt{h^2+k^2+hk}} \quad\ldots(93)$$

Form Symbols. The four-index Bravais symbols are used throughout. When the lattice mode is rhombohedral, Millerian three index symbols are also given. Bravais $hk\bar{i}l$ symbols may be transformed to Millerian hkl symbols by the use of the following formula:

$$\text{Bravais to Miller: } \frac{1}{3}\ 0\ \frac{\bar{1}}{3}\ \frac{1}{3} \Big/ \frac{\bar{1}}{3}\ \frac{1}{3}\ 0\ \frac{1}{3} \Big/ 0\ \frac{\bar{1}}{3}\ \frac{1}{3}\ \frac{1}{3} \quad\ldots\ldots(94)$$

The reverse transformation is as follows:

$$\text{Miller to Bravais: } 1\ \bar{1}\ 0 \ /\ 0\ 1\ \bar{1} \ /\ \bar{1}\ 0\ 1 \ /\ 1\ 1\ 1 \quad\ldots\ldots\ldots\ldots(95)$$

Indexing of Forms. In the indexing of forms in the hexagonal system, two coordinate axes, P_1 and P_2 (Fig. 14), are used which are normal to $(10\bar{1}0)$ and $(01\bar{1}0)$, respectively (see also Fig. 15). The positive unit

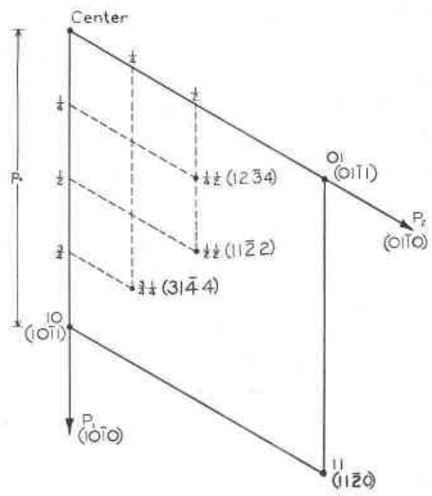

FIG. 14. Determination of Bravais Form Indices from the Gnomonic Projection in the Hexagonal System.

lengths along these coordinate axes extend from the center of projection to the gnomonic poles of $(10\bar{1}1)$ and $(01\bar{1}1)$. The Bravais indices $(hk\bar{i}l)$ are found as follows:

$h=P_1$ coordinate

$k=P_2$ coordinate

$i=(h+k)$

$l=1$

(clearing fractions, if necessary)

In figure 14 the faces for which P_1 and P_2 coordinates are given receive the following indices:

$$\frac{1}{4}\,\frac{1}{2} = \left(\frac{1}{4}\cdot\frac{2}{4}\cdot\frac{\bar{3}}{4}\cdot 1\right) = (12\bar{3}4)$$

$$\frac{1}{2}\,\frac{1}{2} = \left(\frac{1}{2}\cdot\frac{1}{2}\cdot\bar{1}\cdot 1\right) = (11\bar{2}2)$$

$$\frac{3}{4}\,\frac{1}{4} = \left(\frac{3}{4}\cdot\frac{1}{4}\cdot\bar{1}\cdot 1\right) = (31\bar{4}4)$$

Order of Form Listing. The order of listing in the hexagonal system is given in Tables 7*a*, 7*b*, and 8. Table 7*a* includes those symmetry classes in which the lattice mode is necessarily primitive hexagonal (P). Table 7*b* is the order of listing for the remaining classes if their lattice mode is primitive hexagonal. If, however, the lattice mode is rhombohedral (*R*), the order of listing for these same classes is given in Table 8. The division

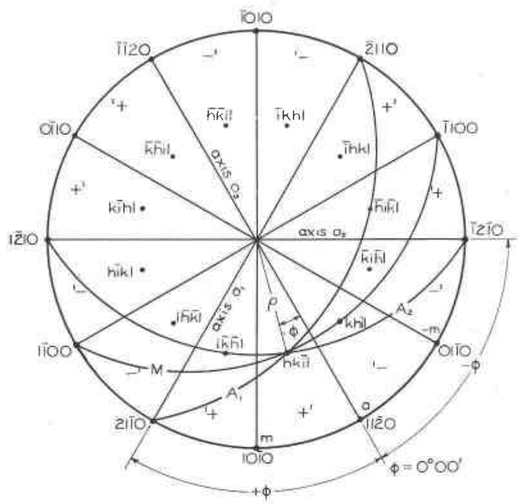

FIG. 15. Stereographic Projection Showing Form Designation and Angles in the Hexagonal System.

into three tables is necessary since certain forms which are complementary (or correlative) merohedral forms in the primitive lattice mode are actually holohedral in the rhombohedral lattice mode. For example, $0h\bar{h}l$ forms must be listed with their complementary $h0\bar{h}l$ forms if the lattice is primitive; but they are listed separately if the lattice is rhombohedral. The letters *c*, *m*, *a* are reserved for the faces (0001), $(10\bar{1}0)$, $(11\bar{2}0)$, respectively. The plus and minus 30° meridians define right and left forms (Fig. 15).

Table 7*a*. Order of Form Listing in Classes with Lattice Mode Uniquely Hexagonal—*P*

Class $\bar{6}$	6		$\frac{6}{m}$	$\bar{6}\,m\,2$	$6\,m\,m$		$6\,2\,2$	$\frac{6}{m}\,\frac{2}{m}\,\frac{2}{m}$
	Lower	Upper			Lower	Upper		
c 0001	x	x	x	x	x	x	x	x
m $10\bar{1}0$		x	x	x		x	x	x
$-m$ $01\bar{1}0$				x				
a' $11\bar{2}0$		x	x	x		x	x	x $(i>h>k)$
$'a$ $2\bar{1}\bar{1}0$								
j' $hk\bar{\imath}0$		x	x	x		x	x	x in order of increasing $\frac{h}{k}$
$'j$ $i\bar{k}\bar{h}0$		x	x					
$-'j$ $kh\bar{\imath}0$				x				
$-j'$ $\bar{k}i\bar{h}0$								
q $h0\bar{h}l$	x	x	x	x	x	x	x	x in order of increasing $\frac{h}{l}$
$-q$ $0h\bar{h}l$				x				
e' $hh\,\overline{2h}\,l$	x	x	x	x	x	x	x	x in order of increasing $\frac{h}{l}$
$'e$ $2h\,\bar{h}\bar{h}\,l$								
s' $hk\bar{\imath}l$	x	x	x	x	x	x	x	x in order of increasing $\frac{h}{k}$;
$'s$ $i\bar{k}\bar{h}l$	x	x					x	in groups of equal $\frac{h}{k}$,
$-'s$ $kh\bar{\imath}l$				x				list in order of increasing
$-s'$ $\bar{k}i\bar{h}l$								$\frac{h}{l}$.

Table 7*b*. Order of Form Listing in Classes with Lattice Mode Not Uniquely Hexagonal—*P*

Class 3		$\bar{3}$	$3\,m$		$3\,2$	$\bar{3}\,\frac{2}{m}$
Lower	Upper		Lower	Upper		
x	c 0001	x	x	x	x	x
	m $10\bar{1}0$	x		x	x	x
	$-m$ $01\bar{1}0$			x		
	a' $11\bar{2}0$	x		x	x	x
	$'a$ $2\bar{1}\bar{1}0$				x	$(i>h>k)$

TABLE 7b. *Continued*

Class 3 Lower		Class 3 Upper	$\bar{3}$	$3m$ Lower	$3m$ Upper	$3\,2$	$\bar{3}\,\frac{2}{m}$
	j'	$hk\bar{i}0$	x		x	x	x in order of increasing $\frac{h}{k}$
	$'j$	$i\bar{k}\bar{h}0$	x			x	
	$-'j$	$kh\bar{i}0$	x		x		
	$-j'$	$\bar{k}i\bar{h}0$	x				
x	q	$h0\bar{h}l$	x	x	x	x	x in order of increasing $\frac{h}{l}$
x	Q	$0h\bar{h}l$	x	x	x	x	x
x	e'	$hh\overline{2h}\,l$	x	x	x	x	x same as for $h0\bar{h}l$
x	$'e$	$2h\,\bar{h}\bar{h}\,l$	x			x	x
x	s'	$hk\bar{i}l$	x	x	x	x	x in order of increasing $\frac{h}{k}$;
x	$'s$	$i\bar{k}\bar{h}l$	x			x	in groups of equal $\frac{h}{k}$,
x	$-'s$	$kh\bar{i}l$	x	x	x	x	x list in order of increasing $\frac{h}{l}$.
x	$-s'$	$\bar{k}i\bar{h}l$	x			x	

TABLE 8. ORDER OF FORM LISTING IN CLASSES WITH LATTICE MODE NOT UNIQUELY HEXAGONAL—R

Class 3 Lower		Class 3 Upper	$\bar{3}$	$3m$ Lower	$3m$ Upper	$3\,2$	$\bar{3}\,\frac{2}{m}$
x	c	0001	x	x	x	x	x
	m	$10\bar{1}0$	x		x	x	x
	$-m$	$01\bar{1}0$			x		
	a'	$11\bar{2}0$	x		x	x	x
	$'a$	$2\bar{1}\bar{1}0$				x	$(i>h>k)$
	j'	$hk\bar{i}0$	x		x	x	x in order of increasing $\frac{h}{k}$
	$'j$	$i\bar{k}\bar{h}0$	x			x	
	$-'j$	$kh\bar{i}0$	x		x		
	$-j'$	$\bar{k}i\bar{h}0$	x				
x	q	$h0\bar{h}l$	x	x	x	x	x in order of increasing $\frac{h}{l}$

TABLE 8. *Continued*

Class 3			$\bar{3}$	$3\,m$		$3\,2$	$\bar{3}\,\frac{2}{m}$
Lower	Upper			Lower	Upper		
x	Q	$0h\bar{h}l$	x	x	x	x	x same as for $h0\bar{h}l$
x	e'	$hh\,\bar{2h}\,l$	x	x	x	x	x same as for $h0\bar{h}l$
x	$'e$	$2h\,\bar{h}\bar{h}\,l$	x			x	
x	s'	$hk\bar{i}l$	x	x	x	x	x in order of increasing $\frac{h}{k}$;
x	$'s$	$i\bar{k}\bar{h}l$	x			x	in groups of equal $\frac{h}{k}$,
							list in order of increasing $\frac{h}{l}$.
x	$-'s$	$k\bar{h}\bar{i}l$	x	x	x	x	x same as for $hk\bar{i}l$
x	$-s'$	$\bar{k}i\bar{h}l$	x			x	

Hexagonal System Angles. The angles given in the hexagonal system are: (Fig. 15) the azimuth angle ϕ with ϕ of $(11\bar{2}0)$ 0°00′; the polar angle ρ, equal to the interfacial angle to c (0001); M, the interfacial angle to $(1\bar{1}00)$, given for crystals of the primitive mode only; A_1, the interfacial angle to $(2\bar{1}\bar{1}0)$, given for crystals of the rhombohedral mode only; and A_2, the interfacial angle to $(\bar{1}2\bar{1}0)$. The following formulae are useful in calculating these angles.

$$\tan\phi=\frac{h-k}{h+k}\cdot\frac{1}{\sqrt{3}}\text{ or }\cot\phi=\frac{h+k}{h-k}\cdot\sqrt{3}\quad\ldots\ldots\ldots(96)$$

$$\text{for }(h0\bar{h}l)\text{ and }(0h\bar{h}l)\ \tan\rho=\frac{p_0h}{l}\quad\ldots\ldots\ldots(97)$$

$$\text{for }(hh\,\bar{2h}\,l)\ \tan\rho=\frac{p_0h\sqrt{3}}{l}$$

$$\text{for }(hk\bar{i}l)\ \tan\rho=\frac{p_0}{l}\sqrt{h^2+k^2+hk}$$

TABLE 9. CALCULATION OF INTERFACIAL ANGLES IN THE HEXAGONAL SYSTEM (FIG. 15)

Form	M	A_1	A_2
$h0\bar{h}l$	$\cos M = \frac{\sin\rho}{2}$	$\cos A_1 = \frac{\sqrt{3}}{2}\sin\rho$	$90°00'$
$0h\bar{h}l$	$\cos(180° - M) = \frac{\sin\rho}{2}$	$90°00'$	$\cos A_2 = \frac{\sqrt{3}}{2}\sin\rho$
$hh\,2\bar{h}\,l$	$90°00'$	$\cos A_1 = \frac{\sin\rho}{2}$	$\cos A_2 = \frac{\sin\rho}{2}$
$2h\,\bar{h}\bar{h}\,l$	$\cos M = \frac{\sqrt{3}}{2}\sin\rho$	$A_1 = 90°00' - \rho$	$\cos(180° - A_2) = \frac{\sin\rho}{2}$
$hk\bar{i}l$	$\cos M = \sin\rho\sin\phi$	$\cos A_1 = \sin\rho\cos(60° - \phi)$	$\cos A_2 = \sin\rho\cos(60° + \phi)$
$i\bar{k}\bar{h}l$	$\cos M = \sin\rho\cos(30° + \phi)$	$\cos A_1 = \sin\rho\cos\phi$	$\cos(180° - A_2)$ $= \sin\rho\cos(60° + \phi)$
$kh\bar{i}l$	$\cos(180° - M) = \sin\rho\sin\phi$	$\cos A_1 = \sin\rho\cos(60° + \phi)$	$\cos A_2 = \sin\rho\cos(60° - \phi)$
$\bar{k}i\bar{h}l$	$\cos(180° - M)$ $= \sin\rho\cos(30° + \phi)$	$\cos(180° - A_1)$ $= \sin\rho\cdot\cos(60° + \phi)$	$\cos A_2 = \sin\rho\cos\phi$

The letter ϕ in the above table always refers to the plus right position of the form.

Since ϕ is a constant for each h/k value in the hexagonal system, the following table of ϕ angles is given with variations in h and k from 1 to 11. The angles are given according to increasing magnitude to facilitate rapid comparison with measured values.

TABLE 10. PHI ANGLES IN THE HEXAGONAL SYSTEM

$h:k$ ϕ	$h:k$ ϕ	$h:k$ ϕ
1: 1—0°00′00″	3:2— 6°35′12″	3:1—16°06′08″
11:10—1 34 29	11:7— 7 18 40	10:3—17 16 10
10: 9—1 44 26	8:5— 7 35 21	7:2—17 47 01
9: 8—1 56 42	5:3— 8 12 48	11:3—18 15 30
8: 7—2 12 15	7:4— 8 56 54	4:1—19 06 24
7: 6—2 32 35	9:5— 9 22 01	9:2—20 10 25
6: 5—3 00 16	11:6— 9 38 15	5:1—21 03 06
11: 9—3 18 16	2:1—10 53 36	11:2—21 47 12
5: 4—3 40 14	11:5—12 12 59	6:1—22 24 23
9: 7—4 07 40	9:4—12 31 11	7:1—23 24 48
4: 3—4 42 54	7:3—13 00 14	8:1—24 10 57
11: 8—5 12 31	5:2—13 53 52	9:1—24 47 29
7: 5—5 29 47	8:3—14 42 17	10:1—25 17 06
10: 7—5 49 03	11:4—15 04 45	11:1—25 41 36

Special Relations in Rhombohedral Lattices. In crystals with a rhombohedral lattice mode there are generally given, in addition to the regular hexagonal elements, the following constants:

a_{rh}, the absolute length of the rhombohedral edge, derived from x-ray measurements.

α, the interaxial angle of the direct rhombohedral lattice.

λ, the interaxial angle of the reciprocal rhombohedral lattice.

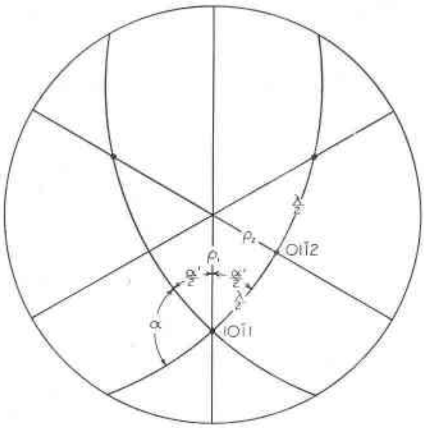

FIG. 16. Stereographic projection of Linear and Polar Interaxial Angles in the Rhombohedral Lattice Mode.

Some formulae relating the hexagonal and rhombohedral lattice modes follow (Fig. 16). If:

a_0 = absolute length along a-axis of hexagonal lattice

c_0 = absolute length along c-axis of hexagonal lattice

$\rho_1 = \rho$ of $(10\bar{1}1) = (0001) \wedge (10\bar{1}1)$

$\rho_2 = \rho$ of $(01\bar{1}2) = (0001) \wedge (01\bar{1}2)$

Then:

$$\sin\frac{\lambda}{2} = \frac{\sqrt{3}}{2}\sin\rho_1 = \sqrt{3}\sin\rho_2 \quad \text{(98)}$$

$$\cot\frac{\alpha'}{2} = \sqrt{3}\cos\rho_1 \quad \text{(99)}$$

primed values are supplements of unprimed

$$\cos\frac{\alpha'}{2} = \frac{\sqrt{3}}{2}\cos\rho_1 \quad \text{(100)}$$

$$\sin\frac{\alpha}{2} = \frac{a_0}{2a_{rh}} = \frac{3a_0}{2\sqrt{3a_0^2 + c_0^2}} = \frac{3}{2\sqrt{3+\left(\frac{c}{a}\right)^2}} \quad \text{(101)}$$

$$a_{rh} = \frac{1}{3}\sqrt{3a_0^2 + c_0^2} \quad \text{(102)}$$

$$\frac{c_0}{a_0}=\sqrt{\left(\frac{3}{2\sin\cdot\frac{\alpha}{2}}\right)^2-3} \quad (103)$$

$$\text{Volume rhombohedral cell}=\frac{a_0^2\ c_0\sqrt{3}}{6} \quad (104)$$

Isometric System

Elements and angle tables are not given for isometric minerals, since the same angular relations hold for all. Angles for commonly found isometric forms are listed in Table 11. The meaning of the angles given is shown in Fig. 17.

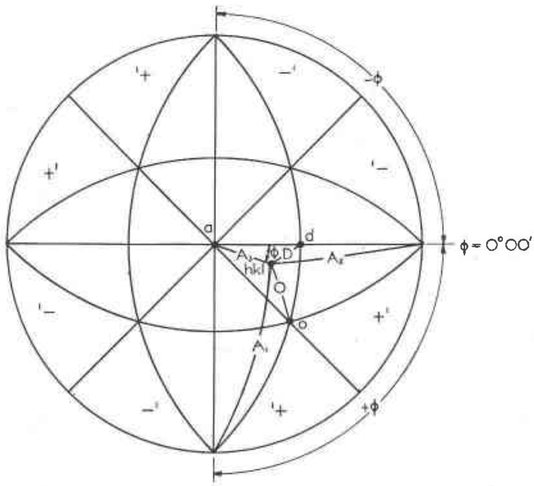

FIG. 17. Stereographic Projection Showing Form Designation and Angles in the Isometric System.

ϕ = azimuth angle measured from (010), zero meridian
$\rho=A_3$ = interfacial angle with (001)
A_1 = interfacial angle with (100)
A_2 = interfacial angle with (010)
D = interfacial angle with (011)
O = interfacial angle with (111)

Forms for which the same letters are used in all species are given letters in Table 11. In order that there may be sufficient remaining letters to designate the form assemblage of any one species, no convention has been adopted for the lettering of other forms.

TABLE 11. ISOMETRIC ANGLE TABLE

	Form	ϕ	$\rho=A_3$	A_1	A_2	D	O
a	001	—	0°00′	90°00′	90°00′	45°00′	54°44′
d	011	0°00′	45 00	90 00	45 00	0 00	35 16
o	111	45 00	54 44	54 44	54 44	35 16	0 00
	0·1·15	0 00	3 49	90 00	86 11	41 11	52 05
	0·1·10	0 00	5 42½	90 00	84 17½	39 17½	50 48½
	018	0 00	7 07½	90 00	82 52½	37 52½	49 52½
	017	0 00	8 08	90 00	81 52	36 52	49 13
	016	0 00	9 27½	90 00	80 32½	35 32½	48 22
	015	0 00	11 18½	90 00	78 41½	33 41½	47 12½
	029	0 00	12 31½	90 00	77 28½	32 28½	46 27½
h	014	0 00	14 02	90 00	75 58	30 58	45 33½
	072	0 00	15 56½	90 00	74 03½	29 03½	44 27½
	0·3·10	0 00	16 42	90 00	73 18	28 18	44 02
f	013	0 00	18 26	90 00	71 34	26 34	43 05½
k	025	0 00	21 48	90 00	68 12	23 12	41 22
	037	0 00	23 12	90 00	66 48	21 48	40 42
	049	0 00	23 57½	90 00	66 02½	21 02½	40 21
e	012	0 00	26 33	90 00	63 27	18 27	39 14½
	059	0 00	29 03½	90 00	60 56½	15 56½	38 16½
	047	0 00	29 44½	90 00	60 15½	15 15½	38 01½
l	035	0 00	30 58	90 00	59 02	14 02	37 37
g	023	0 00	33 41½	90 00	56 18½	11 18½	36 48½
	057	0 00	35 32½	90 00	54 27½	9 27½	36 21
ϑ	034	0 00	36 52	90 00	53 08	8 08	36 04
	045	0 00	38 39½	90 00	51 20½	6 20½	35 45½
	078	0 00	41 11	90 00	48 49	3 49	35 26½
	1·1·12	45 00	6 43½	85 15	85 15	40 28	48 00½
	1·1·11	45 00	7 19½	84 49½	84 49½	41 14½	47 24½
	1·1·10	45 00	8 03	84 19	84 19	39 38	46 41
	119	45 00	8 56	83 42	83 42	39 05½	45 48
	118	45 00	10 01½	82 56	82 56	38 26	44 42½
	117	45 00	11 25½	81 57	81 57	37 37	43 18½
	116	45 00	13 16	80 40	80 40	36 35	41 28

TABLE 11. *Continued*

	Form	ϕ	$\rho=A_3$	A_1	A_2	D	O
	115	45°00′	15°47½′	78°54½′	78°54½′	35°16′	38°56½′
	114	45 00	19 28½	76 22	76 22	33 33½	35 15½
	227	45 00	22 00	74 38½	74 38½	32 33	32 44
m	113	45 00	25 14½	72 27	72 27	31 29	29 29½
	338	45 00	27 56½	70 39	70 39	30 48	26 47½
	225	45 00	29 30	69 37½	69 37½	30 30	25 14
	337	45 00	31 13	68 33	68 33	30 15	23 31
n	112	45 00	35 16	65 54½	65 54½	30 00	19 28
	447	45 00	38 56½	63 36½	63 36½	30 12	15 47½
	335	45 00	40 19	62 46½	62 46½	30 23	14 25
β	223	45 00	43 19	60 59	60 59	30 58	11 25
	334	45 00	46 41	59 02½	59 02½	31 54½	8 03
	556	45 00	49 41	57 22½	57 22½	32 59½	5 03
	188	7 07½	45 13½	84 56	45 13½	5 04	30 12
	177	8 08	45 17½	84 14	45 17½	5 46	29 30
	166	9 27½	45 23½	83 16½	45 23½	6 43½	28 32½
	155	11 18½	45 33½	81 57	45 33½	8 03	27 13
ρ	144	14 02	45 52	79 58½	45 52	10 01½	25 14½
	277	15 56½	46 07½	78 34½	46 07½	11 25½	23 50½
q	133	18 26	46 30½	76 44	46 30½	13 16	22 00
	255	21 48	47 07½	74 12½	47 07½	15 47½	19 28½
p	122	26 34	48 11½	70 31½	48 11½	19 28½	15 47½
	355	30 58	49 23	67 00½	49 23	22 59½	12 16½
r	233	33 41½	50 14½	64 45½	50 14½	25 14½	10 01½
	344	36 52	51 20½	62 03½	51 20½	27 56½	7 19½
	455	38 39½	52 01	60 30	52 01	29 30	5 46
	1·6·12	9 27½	26 53	85 44	63 31	18 54½	35 22½
	1·6·11	9 27½	28 56½	85 26	61 29½	17 00½	34 14
	1·5·10	11 18½	27 01	84 53½	63 33	19 06½	34 37
	1·6·10	9 27½	31 18½	85 06	59 09½	14 51	33 01
	179	8 08	38 09½	84 59½	52 17½	8 42	30 57½
	128	26 34	15 37	83 05	76 04	31 39	40 08
	138	18 26	21 34	83 19½	69 35½	25 17	36 21

TABLE 11. *Continued*

	Form	ϕ	$\rho = A_3$	A_1	A_2	D	O
	148	14°02′	27°16′	83°37′	63°36½′	19°28½′	33°29½′
	158	11 18½	32 31	83 57	58 11½	14 18½	31 34½
	127	26 34	17 43	82 10½	74 12½	30 00	38 13
	137	18 26	24 18½	82 31	67 00½	22 59½	34 13½
	157	11 18½	36 04	83 22	54 44	11 32	29 56
	2·3·12	33 41½	16 43½	80 49	76 09	32 10	38 26
	156	11 18½	40 21½	82 42	50 35	8 57	28 22½
	2·5·11	21 48	26 05	80 36	65 54½	22 31	31 57
	125	26 34	24 05½	79 29	68 35	25 21	32 35½
	135	18 26	32 18½	80 16	59 32	17 01½	28 33½
	145	14 02	39 30½	81 07½	51 53	10 54	27 01
	239	33 41½	21 50	78 05½	71 58½	28 56	33 31
	249	26 34	26 25½	78 31½	66 33	23 50½	30 29½
	269	18 26	35 06	79 31½	56 56½	15 22	26 50½
	238	33 41½	24 15½	76 49½	70 00½	27 34½	31 12
t	124	26 34	29 12½	77 23½	64 07½	22 12½	28 07½
	134	18 26	38 19½	78 41½	53 57½	13 54	25 04
	3·5·11	30 58	27 55½	76 03½	66 19½	24 40	28 13½
	237	33 41½	27 15	75 17	67 36½	26 06	28 22½
	247	26 34	32 34½	76 04	61 13	20 33	25 22
	3·4·10	36 52	26 34	74 26	69 02	27 41½	28 37
	3·5·10	30 58	30 15	74 59	64 24½	23 37	26 08
	3·7·10	23 12	37 17½	76 11½	56 09½	17 00	23 16½
	236	33 41½	31 00	73 24	64 37½	24 37	24 52
s	123	26 34	36 42	74 30	57 41½	19 06½	22 12½
	358	30 58	36 05	72 21½	59 40	21 47	21 04
	235	33 41½	35 47½	71 04	60 52½	23 24½	20 31
	458	38 39½	38 40½	67 01½	60 47½	26 12	16 42
	346	36 52	39 48½	67 24½	59 11½	25 07½	16 03½
	234	33 41½	42 02	68 12	56 08½	23 12	15 13½
	345	36 52	45 00	64 54	55 33	25 50½	11 33½
	578	35 32½	47 04½	64 48½	53 25½	25 27½	10 36
	456	38 39½	46 47½	62 54½	55 18½	27 33½	9 19½

The order of form listing is given in Table 12 for the five isometric symmetry classes.

Calculation of Angles. The angles in Table 11 are calculated as follows:

$$\tan \phi = \frac{h}{k} \text{ (see Table 7)} \qquad (105)$$

$$\tan \rho = \frac{h}{l \sin \phi} = \frac{k}{l \cos \phi} \qquad (106)$$

$$\cos A_1 = \sin \rho \sin \phi \qquad (107)$$

$$\cos A_2 = \sin \rho \cos \phi \qquad (108)$$

$$\cos D = \frac{\sqrt{2}}{2}(\cos \rho + \sin \rho \cos \phi) \qquad (109)$$

$$\cos O = \cos \rho \cos 54°44' + \sin \rho \sin 54°44' \cos (45° - \phi) \qquad (110)$$

TABLE 12. ORDER OF FORM LISTING IN THE ISOMETRIC SYSTEM

Class	2 3	$\frac{2}{m}\bar{3}$	$\bar{4}\ 3\ m$	4 3 2	$\frac{4}{m}\bar{3}\frac{2}{m}$	
a	001	*x*	*x*	*x*	*x*	
d	011	*x*	*x*	*x*	*x*	$(h<k<l)$
o	111	*x*	*x*	*x*	*x*	
$-o$	$\bar{1}11$		*x*			
e′	$0kl$	*x*	*x*	*x*	*x*	in order of increasing $\frac{k}{l}$
′e	$k0l$	*x*				
j	hhl	*x*	*x*	*x*	*x*	in order of increasing $\frac{h}{l}$
$-j$	$\bar{h}hl$		*x*			
n	hll	*x*	*x*	*x*	*x*	in order of increasing $\frac{h}{l}$
$-n$	$\bar{h}ll$		*x*			
s′	hkl	*x*	*x*	*x*	*x*	in order of increasing $\frac{h}{l}$;
′s	khl	*x*				in groups of equal $\frac{h}{l}$,
$-'s$	$\bar{h}kl$		*x*	*x*		list in order of increasing $\frac{k}{l}$.
$-s'$	$\bar{k}hl$					

Acknowledgments

Valuable suggestions and criticisms relative to the manuscript have been given by Professors Charles Palache and Harry Berman of Harvard University and by Professor J. D. H. Donnay of Laval University. Professor M. A. Peacock of the University of Toronto has been of considerable help in the formulation of some of the procedures described in this paper. I wish to express my appreciation, here, for the assistance given by each of these men. The figures have been drawn from my sketches by Howard T. Evans.