NOMOGRAMS OF OPTIC ANGLE FORMULAE*

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ABSTRACT

Nomograms are presented herewith for the solution of several well-known petrographic formulae that are used in the measurement of optic angles. To solve for any one of the four variables α, β, γ, and V, when three of them are known, a grid type nomogram has been prepared; and a similar chart has been drawn to show the relationships between the variables α, β, γ, and E. Two nomograms are also presented for the solution of the equations $\sin E=\beta \sin V$ and $D=K \sin E$ (Mallard's formula). The method of preparing these charts is outlined, and their uses given. The topic of errors in observations and in computations is also considered.

INTRODUCTION

The equations that give the true and apparent optic angles of a biaxial mineral, as functions of its three indices of refraction, are well known to petrographers. But these equations contain four variables, and cannot be represented by single graphs, using ordinary two-dimensional Cartesian or polar coordinates. A graphic solution of an equation that relates approximately the variables V, α, β, and γ was published by Wright, who utilized a combination of Cartesian and polar coordinates; but the use of this chart requires a preliminary subtraction of α from β, and of α from γ, in order to obtain an equation containing only three variables. Later Wright published three similar graphs, charting both the approximate and true equations, but these likewise required preliminary arithmetical work between the variables α, β, and γ, and were therefore essentially three variable charts. In more recent years, Smith published a graph for obtaining the value of V, when α, β and γ are known, charting an approximation formula similar to the one charted by Wright. This graph uses one or more moving calibrated scales, and requires a threefold visual interpolation. Still later, Lane and Smith published another graph of the same equation, utilizing a different method of charting, but still retaining the moving calibrated scale. In the present paper, the true equations relating α, β, γ, and V and α, β, γ, and E are charted as nomograms of the grid type. Such graphs require for their use only a single uncalibrated straight-line index, and involve the least possible degree of visual interpolation.

Two other equations, containing only three variables, are also much used in the measurement of optic angles. These are the equations $\sin E=\beta \sin V$, and $D=K \sin E$ (Mallard's formula). The first of these was originally charted by Fedorov, using a combination of Cartesian and

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polar coordinates; and subsequently Wright published two graphs of these equations, one of which was essentially a reprint of Fedorov's chart, whereas the second was a graph in Cartesian coordinates. Still later, Wright published three additional graphs of this equation, charted by the same methods. Mallard's equation, on the other hand, was first charted in Cartesian coordinates by Becke, and later in a combination of Cartesian and polar coordinates by de Souza-Brandão. A special slide-rule was also constructed for its solution by Schwarzmann. Doubtless many others have likewise prepared graphs of these two simple equations. But excepting Schwarzmann's slide-rule, which is not readily available, all of these solutions require visual interpolation whereas a three-variable nomogram is free from this drawback. None of these graphs, therefore, can be used as easily and quickly as a nomogram.

The equation $\sin E=\beta \sin V$ can readily be transformed, by a suitable choice of parameters, into Mallard's formula, and vice versa. Hence both equations convey essentially the same information, and it would be entirely feasible to chart them in a single three-line nomogram, using double calibrations on two of the scales. For the sake of clarity, however, the two equations have been separately charted.

Optic Angle Equations

Six equations are commonly given, relating the four variables α, β, γ, and V, but it is necessary to consider only two of these, both of which can be presented together in a single nomogram. The selected equations are as follows:

$$\sin^2 V_\gamma=\frac{\dfrac{1}{\alpha^2}-\dfrac{1}{\beta^2}}{\dfrac{1}{\alpha^2}-\dfrac{1}{\gamma^2}} \tag{1}$$

$$\sin^2 V_\alpha=\frac{\dfrac{1}{\beta^2}-\dfrac{1}{\gamma^2}}{\dfrac{1}{\alpha^2}-\dfrac{1}{\gamma^2}} \tag{2}$$

where V_γ = one half the true optic angle of a positive biaxial mineral.
V_α = one half the true optic angle of a negative biaxial mineral.
α = the least index of refraction.
β = the intermediate index of refraction.
γ = the greatest index of refraction.

We have also the following equations:

$$\sin E=\beta \sin V \tag{3}$$

$$D=K \sin E \tag{4}$$

where E = one half the apparent optic angle of a biaxial mineral, as measured in air.

D = a distance measured on a micrometer scale, usually in the lower focal plane of the ocular.

K = Mallard's constant, as experimentally determined.

Combinbing equations (1) with (3), and (2) with (3), we obtain the following equations, that are likewise to be charted:

$$\frac{\sin^2 E_\gamma}{\beta^2}=\frac{\dfrac{1}{\alpha^2}-\dfrac{1}{\beta^2}}{\dfrac{1}{\alpha^2}-\dfrac{1}{\gamma^2}} \tag{5}$$

$$\frac{\sin^2 E_\alpha}{\beta^2}=\frac{\dfrac{1}{\beta^2}-\dfrac{1}{\gamma^2}}{\dfrac{1}{\alpha^2}-\dfrac{1}{\gamma^2}} \tag{6}$$

Eliminating E from equations (3) and (4), another very useful equation is obtained, namely,

$$D=K\beta \sin V \tag{7}$$

Combining equations (1) with (7), and (2) with (7), still another useful pair of equations are made available, as follows:

$$\sin^2 V_\gamma=\frac{D^2\gamma^2}{K^2\alpha^2\gamma^2+D^2(\gamma^2-\alpha^2)} \tag{8}$$

$$\sin^2 V_\alpha=\frac{D^2\alpha^2}{K^2\alpha^2\gamma^2-D^2(\gamma^2-\alpha^2)} \tag{9}$$

For reasons later stated, nomograms of formulae (7), (8) and (9) have not been prepared

Preparation of Charts

In equation (1), let $\sin^2 V_\gamma$ be temporarily a constant, that is, let $\sin^2 V_\gamma=C$. We then have

$$(1-C).\frac{1}{\alpha^2}+C.\frac{1}{\gamma^2}-\frac{1}{\beta^2}=0$$

It is now desired to introduce into the above equation the scale moduli of the nomogram, and also to write the equation in the form of a vanishing determinant. As the values of α and γ may be nearly equal in some biaxial minerals, the range of these two variables, and also of β, may be taken as identical, and their scale moduli may be represented by the same constant, M. The transformation to a vanishing determinant is accomplished by writing two auxiliary equations in new variables, in such a manner as to produce three equations with common roots. The coefficients of these three equations will then yield the desired determinant.

Thus let $x=M.\frac{1}{\alpha^2}$, and $y=M.\frac{1}{\gamma^2}$

This change in variables yields the following equations:

$$\frac{1-C}{M}.x+\frac{C}{M}.y-\frac{1}{\beta^2}=0$$

$$x+0y-M.\frac{1}{\alpha^2}=0$$

$$0x+y-M.\frac{1}{\gamma^2}=0$$

The resulting determinant is as follows:

$$\begin{vmatrix} \frac{1-C}{M} & \frac{C}{M} & -\frac{1}{\beta^2} \\ 1 & 0 & -M.\frac{1}{\alpha^2} \\ 0 & 1 & -M.\frac{1}{\gamma^2} \end{vmatrix} =0$$

Now the index line of a nomogram must connect certain values of α, β, and γ on three scales, in such a manner as to satisfy the equations of these three variables. In other words, three points on each of the three graduated scales must lie on a straight line; and for this condition to exist, all the elements of one row or one column of a vanishing determinant must equal unity. The above determinant is readily changed into this form, by the usual methods of transformation, giving the following:

$$\begin{vmatrix} 0 & M.\frac{1}{\alpha^2} & 1 \\ C & M.\frac{1}{\beta^2} & 1 \\ 1 & M.\frac{1}{\gamma^2} & 1 \end{vmatrix} =0$$

The dimensions of the chart will be taken as S inches square, and since this chart is plotted by means of rectangular coordinates, the maximum X dimension equals S. We introduce S into the vanishing determinant merely by multiplying it into each element of the first column. The maximum Y dimension is determined by the relationship: $M=\frac{S}{\frac{1}{n_2^2}-\frac{1}{n_1^2}}$, where

where n_1 and n_2 are respectively the least and greatest value of the indices of refraction, as shown in the chart. The final constructional determinant is then as follows:

$$\begin{vmatrix} 0 & M.\dfrac{1}{\alpha^2} & 1 \\ SC & M.\dfrac{1}{\beta^2} & 1 \\ S & M.\dfrac{1}{\gamma^2} & 1 \end{vmatrix} = 0 \qquad (10)$$

Similarly, by starting originally with equation (2), the following constructional determinant is obtained:

$$\begin{vmatrix} 0 & M.\dfrac{1}{\alpha^2} & 1 \\ S(1-C) & M.\dfrac{1}{\beta^2} & 1 \\ S & M.\dfrac{1}{\gamma^2} & 1 \end{vmatrix} = 0 \qquad (11)$$

In charting determinants (10), and (11), the question arises as to what limits should be taken for the three indices of refraction, α, β, and γ. To include all the biaxial minerals listed in the Larsen-Berman tables, the α scale should begin at 1.324, to include avogadrite; and the γ scale should extend to 4.046, to include stibnite. Limits of this magnitude are impracticable in a chart of this sort, because it is desired to plot most of the index scales with intervals of .002; and any such range would render this degree of subdivision unreadable, when the chart is reduced to its size of publication. The range of 1.450 to 2.000, which includes nearly 90 per cent of the listed minerals, was finally selected as the best possible compromise. This range excludes fewer minerals of low index of refraction than of high index of refraction, because the optical constants of the former are more readily obtained with ordinary petrographic equipment.

This chart, shown in Plate I, was drawn originally 20 inches square, for reduction to 10 inches square as a laboratory chart, and to about 7 inches square for publication. Hence $S=20$. The value of M was taken as

$$M = \frac{S}{\dfrac{1}{n_2^2} - \dfrac{1}{n_1^2}} = \frac{20}{.47562 - .25} = 88.645$$

The same dimensions and scale moduli were also used in the preparation of Plate II, showing the relations existing between the variables α, β, γ, and E.

In the above exposition, the value of $\sin^2 V_\gamma$ has been assumed to remain constant, representing some fixed value of V. This results in a nomogram of three graduated scales. But if a series of values are now assigned to V, the scale of $\dfrac{1}{\alpha^2}$ and $\dfrac{1}{\gamma^2}$ will remain unchanged, and a series

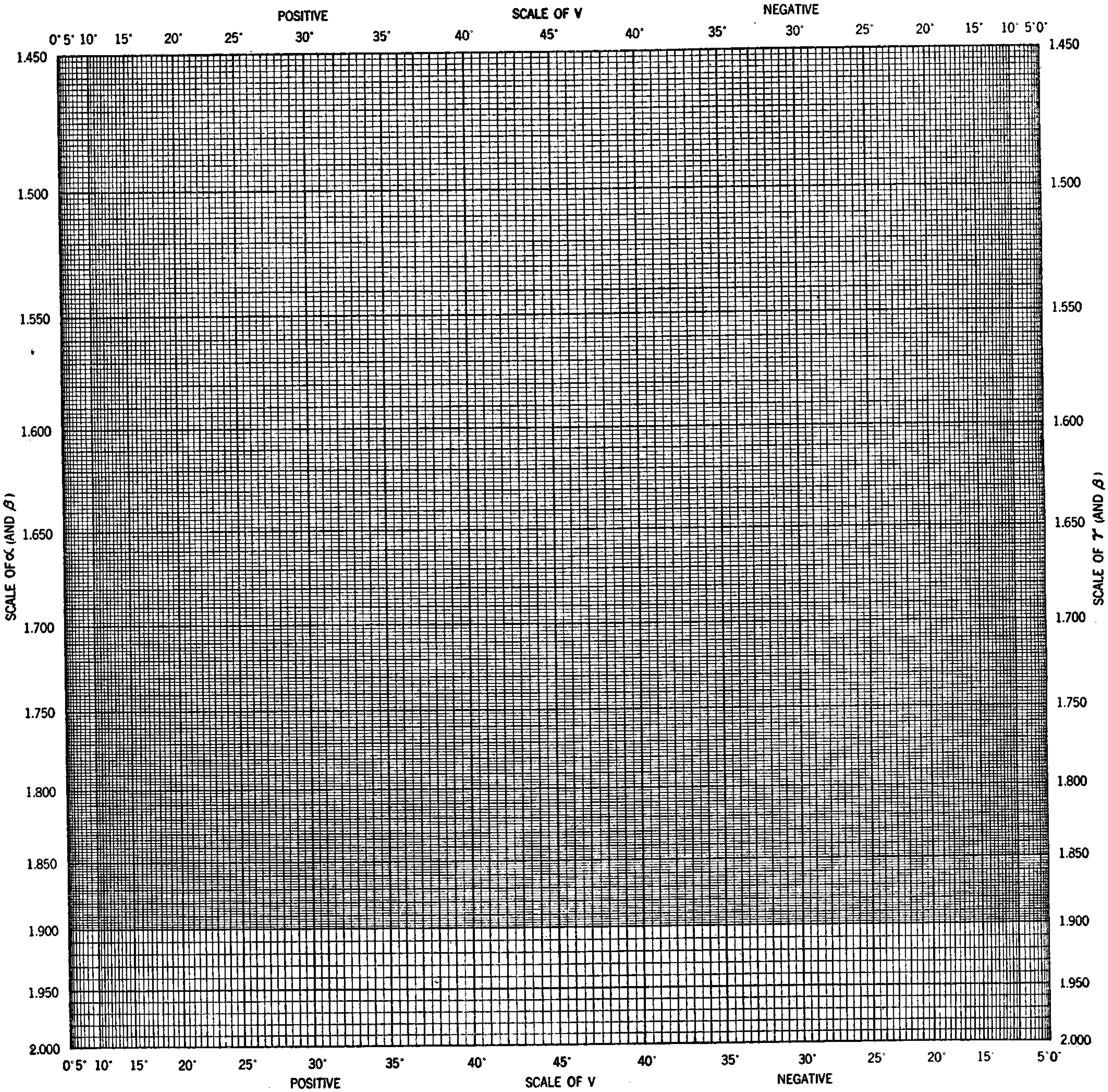

PLATE I. NOMOGRAM OF $\mathrm{SIN}^2 V_\gamma = \dfrac{\frac{1}{\alpha^2} - \frac{1}{\beta^2}}{\frac{1}{\alpha^2} - \frac{1}{\gamma^2}}$ AND $\mathrm{SIN}^2 V_\alpha = \dfrac{\frac{1}{\beta^2} - \frac{1}{\gamma^2}}{\frac{1}{\alpha^2} - \frac{1}{\gamma^2}}$

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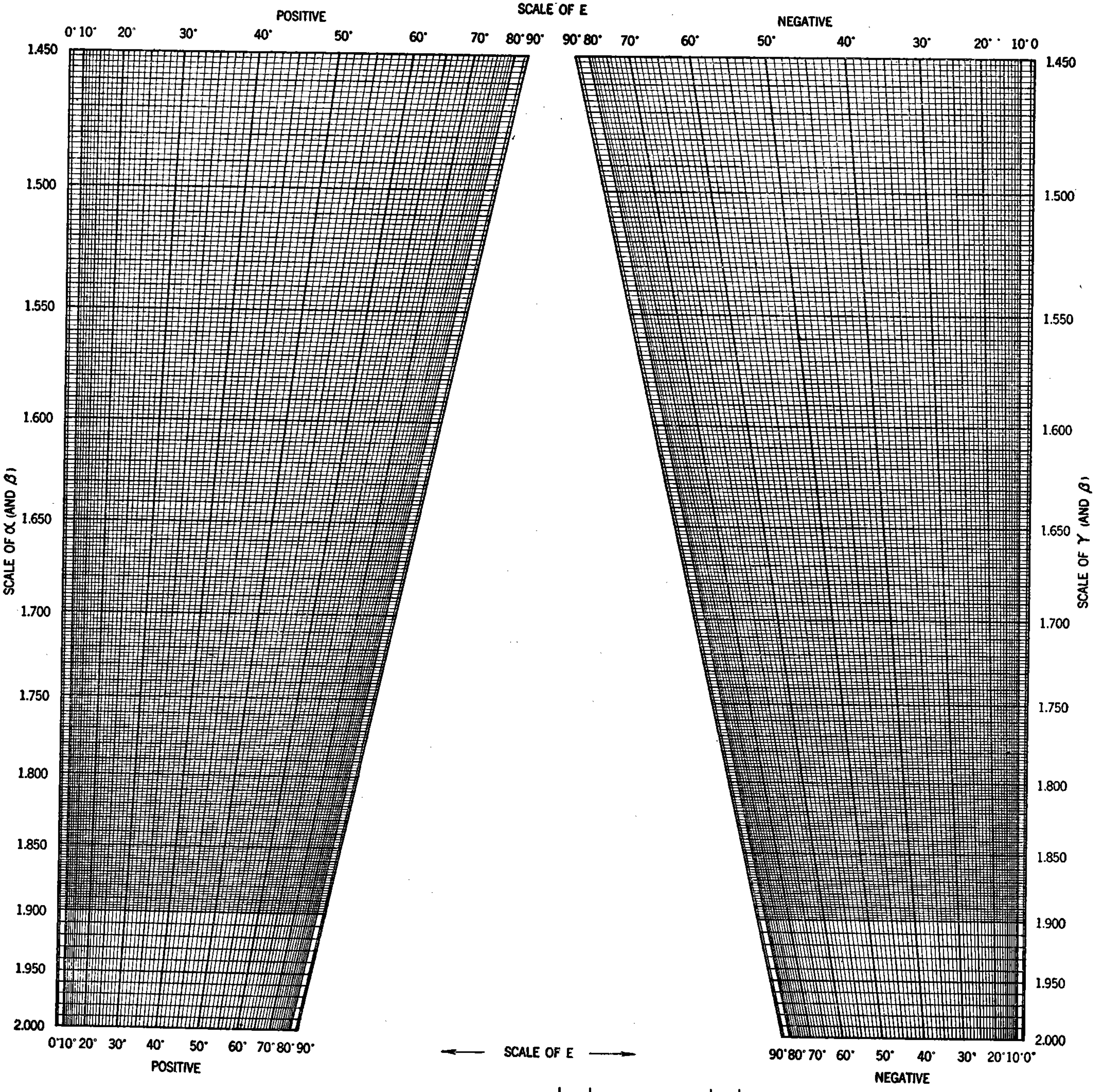

PLATE II. NOMOGRAM OF $\frac{\sin^2 E_\gamma}{\beta^2} = \frac{\frac{1}{\alpha^2}-\frac{1}{\beta^2}}{\frac{1}{\alpha^2}-\frac{1}{\gamma^2}}$ AND $\frac{\sin^2 E_\alpha}{\beta^2} = \frac{\frac{1}{\beta^2}-\frac{1}{\gamma^2}}{\frac{1}{\alpha^2}-\frac{1}{\gamma^2}}$

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of $\frac{1}{\beta^2}$ scales will be produced, each corresponding to some particular value of V. By connecting like values of $\frac{1}{\beta^2}$ on each of these scales, there will result the grid type of nomogram shown in Plate I. Charting the range of V from 0° to 45°, however, this nomogram will be confined to half the space of S inches; and in the other half it is possible to chart determinant (11) without interference. Plate I is therefore a composite grid nomogram, charting determinants (10) and (11).

Considering now equation (5), and representing $\sin^2 E_\gamma$ as temporarily a constant, we have:

$$(\beta^2 - C) . \frac{1}{\alpha^2} + C . \frac{1}{\gamma^2} - 1 = 0$$

Setting $x = M . \frac{1}{\alpha^2}$ and $y = M . \frac{1}{\gamma^2}$, and utilizing the same algebraic technique already explained, we obtain the following vanishing determinant:

$$\begin{vmatrix} \frac{\beta^2 - C}{M} & \frac{C}{M} & -1 \\ 1 & 0 & -M . \frac{1}{\alpha^2} \\ 0 & 1 & -M . \frac{1}{\beta^2} \end{vmatrix} = 0$$

And from this, by transformation and by the introduction of S, is obtained the constructional determinant that is to be charted:

$$\begin{vmatrix} 0 & M . \frac{1}{\alpha^2} & 1 \\ SC . \frac{1}{\beta^2} & M . \frac{1}{\beta^2} & 1 \\ S & M . \frac{1}{\gamma^2} & 1 \end{vmatrix} = 0 \tag{12}$$

Similarly, by starting initially with equation (6), instead of (5), the following constructional determinant is obtained:

$$\begin{vmatrix} 0 & M . \frac{1}{\alpha^2} & 1 \\ S\left(1 - C . \frac{1}{\beta^2}\right) & M . \frac{1}{\beta^2} & 1 \\ S & M . \frac{1}{\gamma^2} & 1 \end{vmatrix} \tag{13}$$

For obvious reasons it is necessary to take for E a range of values twice as great as for V. It has been found feasible to extend the range of E from 0° to 90°, and still to chart determinants (12) and (13) in a single graph, without interference. A composite grid nomogram of this type is shown in Plate II.

In the two preceding charts (Plates I and II), the relationships are shown between α, β, γ, and V, and between α, β, γ, and E. It would also be possible to prepare a third chart, in which β, is absent, and α, γ, V and E are related. The formulae comprising this group of variables are as follows:

$$\frac{1}{\alpha^2}.\cos^2 V_\gamma+\frac{1}{\gamma^2}.\sin^2 V_\gamma=\frac{\sin^2 V_\gamma}{\sin^2 E_\gamma} \tag{14}$$

$$\frac{1}{\alpha^2}.\sin^2 V_\alpha+\frac{1}{\gamma^2}.\cos^2 V_\alpha=\frac{\sin^2 V_\alpha}{\sin^2 E_\alpha} \tag{15}$$

and the constructional determinants corresponding thereto are:

$$\begin{vmatrix} 0 & M.\dfrac{1}{\alpha^2} & 1 \\ S\sin^2 V_\gamma & M.\dfrac{\sin^2 V_\gamma}{\sin^2 E_\gamma} & 1 \\ S & M.\dfrac{1}{\gamma^2} & 1 \end{vmatrix}=0 \tag{16}$$

$$\begin{vmatrix} S & M.\dfrac{1}{\alpha^2} & 1 \\ S\sin^2 V_\alpha & M.\dfrac{\sin^2 V_\alpha}{\sin^2 E_\alpha} & 1 \\ 0 & M.\dfrac{1}{\gamma^2} & 1 \end{vmatrix}=0 \tag{17}$$

From a preliminary charting of these two determinants, it appears that the values of V and E will be represented by a closely spaced gridwork of two systems of lines, that intersect one another at very oblique angles. It is not believed that such a chart would be sufficiently useful to pay for the labor required in its preparation.

Consider now the charting of the next equation:

$$\sin E=\beta \sin V \tag{3}$$

$$\text{Let } x=M\sin E, \qquad \text{and} \qquad y=N\sin V$$

Substituting as before, we get the three following equations:

$$\frac{1}{M}.x-\frac{\beta}{N}.y \quad =0$$

$$x-M\sin E=0$$

$$y-N\sin V=0$$

And from these, the following vanishing determinant is derived:

$$\begin{vmatrix} \dfrac{1}{M} & -\dfrac{\beta}{N} & 0 \\ 1 & 0 & -M \sin E \\ 0 & 1 & -N \sin V \end{vmatrix} = 0$$

This can be transformed, with S and the scale moduli M and N inserted, into the following constructional determinant:

$$\begin{vmatrix} 0 & M \sin E & 1 \\ \dfrac{SM\beta}{M\beta+N} & 0 & 1 \\ S & -N \sin V & 1 \end{vmatrix} = 0 \qquad (18)$$

In this graph, the range of E is twice that of V, and for this reason, with $M=20$, we have

$$N=\frac{S}{\sin V_2-\sin V_1}=\frac{20}{.707}=28.284$$

PLATE III. NOMOGRAM OF SIN E = β SIN V

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Plotting the values of X and Y in rectangular Cartesian coordinates, a chart would result having the form of a rhomboid. To obviate this, and to produce a square chart, the values of X and Y are plotted in oblique Cartesian coordinates, wherein the positive end of the Y axis makes an angle of 45° with the positive end of the X axis. Hence for a chart of 20 inches square, $S=\sqrt{800}=28.284$.

The circumstances which limited the plottable range of indices of refraction, in Plates I and II, do not apply here. Hence a more complete scale of β is charted. Equation (18) is shown in Plate III.

Using the same method as heretofore applied, equation (4) (Mallard's formula) can be transformed into the following constructional determinant:

$$\begin{vmatrix} 0 & M \sin E & 1 \\ \dfrac{SM}{NK+M} & 0 & 1 \\ S & -ND & 1 \end{vmatrix} = 0 \tag{19}$$

In plotting determinant (19), the limits to be taken for the variables D and K are somewhat uncertain. In many petrographic microscopes, the micrometer scale used for axial angle determinations has 100 divisions, though in the Wright model Bausch and Lomb research microscope, the scale includes 130 divisions of 0.1 mm. each. Nevertheless, only a part of this scale is used, and therefore the limits of 0–50, for the half-scale of D, appear to be satisfactory. The range of K (Mallard's constant) was arbitrarily taken as twice that of D, that is, from 0 to 100. In a chart 20 inches square, the value of M is obviously 20; but $N=\dfrac{20}{50}=0.4$. The value of S, using oblique coordinates for plotting, remains 28.284. The nomogram of Mallard's formula is shown in Plate IV.

Reference has already been made to the following equations:

$$D=K\beta \sin V \tag{7}$$

$$\sin^2 V_\gamma = \frac{D^2\gamma^2}{K^2\alpha^2\gamma^2+D^2(\gamma^2-\alpha^2)} \tag{8}$$

$$\sin^2 V_\alpha = \frac{D^2\alpha^2}{K^2\alpha^2\gamma^2-D^2(\gamma^2-\alpha^2)} \tag{9}$$

For all values of K, a grid nomogram would be needed to represent equation (7); and still more complex nomograms would be required to represent equations (8) and (9). On the other hand, for some fixed value of K, equation (7) could be represented by a three-line chart, and equations (8) and (9) by grid nomograms. It seems a waste of time and labor to prepare charts for all values of K, when only one, or at most a few, such

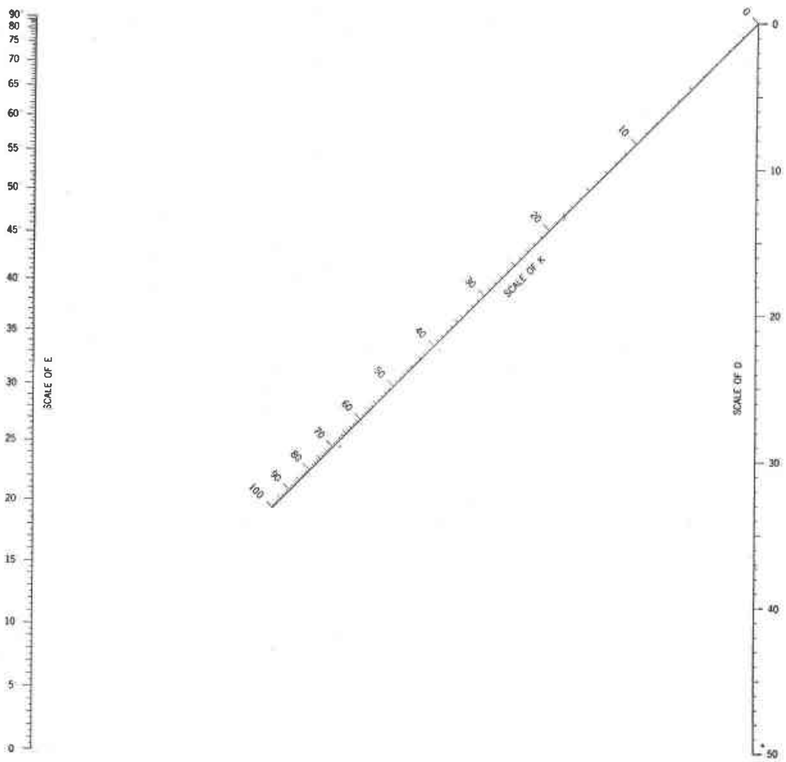

PLATE IV. NOMOGRAM OF D=K SIN E (Mallard's formula)

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values are utilized by any one worker. Hence the preparation of suitable nomograms of these three formulae seems rather to be a project for the individual worker.

Use of Charts

Plates I and II merely represent graphically the relationships that exist between certain functions of α, β, γ, and V (or E); and their use is very simple. All that is really required is a straight-edge, but in actual practice it is better to use a piece of transparent celluloid or zylonite, on the bottom of which has been scratched a straight line. If this line is adjusted on the outer scales, to connect known values of α and γ, it will cut the intervening gridwork at a series of points representing specific values of β and V (or E) that satisfy the charted equations. If the value of β is known, the corresponding value of V (or E) may at once be read from the chart, and vice versa. The optical character of the mineral is likewise indicated. More generally, the value of any one variable can be found, if the values of the other three are known.

Plate I will be used mainly for computing the optic angle of a mineral from its three indices of refraction. But in actual practice, β is more difficult to determine than α and γ. Therefore, Plate II will be of value in determining the value of β, when α, γ, and E are known. The concurrent use of both charts should serve to facilitate the publication of values of α, β, γ, V and E, that accord with one another; and a critical examination of the data that have already been published, will demonstrate the need for this correlation.

Plates III and IV are ordinary N-charts that require little or no explanation. Plate III will probably be used mainly to calculate the value of V, when E and β are known. Plate IV is designed primarily for determining E, from a scale value (D) and Mallard's constant (K). In general, each of these charts shows the relationships existing between three variables.

Errors

It remains to be considered whether these charts yield solutions that have the required degree of accuracy, when considered in terms of unavoidable errors of observation. In this connection, consider formula (1).

$$\sin^2 V_\gamma = \frac{\dfrac{1}{\alpha^2}-\dfrac{1}{\beta^2}}{\dfrac{1}{\alpha^2}-\dfrac{1}{\gamma^2}}\,. \tag{1}$$

Differentiating partially with respect to α, β, and γ, we get:

$$\frac{\partial V_\gamma}{\partial \alpha} = -\frac{1}{\alpha^3\left(\dfrac{1}{\alpha^2}-\dfrac{1}{\gamma^2}\right)}\operatorname{ctn} V_\gamma$$

$$\frac{\partial V_\gamma}{\partial \beta} = \frac{1}{\beta^3\left(\dfrac{1}{\alpha^2}-\dfrac{1}{\gamma^2}\right)}\operatorname{cosec} V_\gamma \sec V_\gamma$$

$$\frac{\partial V_\gamma}{\partial \gamma} = -\frac{1}{\gamma^3\left(\dfrac{1}{\alpha^2}-\dfrac{1}{\gamma^2}\right)}\tan V_\gamma$$

$$\therefore dV_\gamma = \frac{1}{\dfrac{1}{\alpha^2}-\dfrac{1}{\gamma^2}}\left[-\frac{1}{\alpha^3}\operatorname{ctn} V_\gamma\, d\alpha + \frac{1}{\beta^3}\operatorname{cosec} V_\gamma \sec V_\gamma\, d\beta - \frac{1}{\gamma^3}\tan V_\gamma d\gamma\right] \tag{20}$$

Similarly from equation (2), we obtain:

$$dV_\alpha = \frac{1}{\dfrac{1}{\alpha^2}-\dfrac{1}{\gamma^2}}\left[\frac{1}{\alpha^3}\tan V_\alpha\, d\alpha - \frac{1}{\beta^3}\operatorname{cosec} V_\alpha \sec V_\alpha\, d\beta + \frac{1}{\gamma^3}\operatorname{ctn} V_\alpha\, d\gamma\right] \tag{21}$$

Now consider the mineral hornblende, for which $\alpha=1.654$, $\beta=1.666$, $\gamma=1.670$, and V (computed)$=29°49'$. The maximum error will occur when $\Delta\alpha$ and $\Delta\gamma$ are positive and $\Delta\beta$ is negative, or vice versa. Therefore we take $d\alpha=d\gamma=.001$ and $d\beta=-.001$. We then have $\frac{dV_\alpha}{V_\alpha}=.277=28$ per cent, which is approximately the maximum possible error in V. From an examination of formula (21), or by reference to Plate I, it is apparent that this error increases as the birefringence decreases. If such errors as these are present when V is numerically computed, then certainly the graphic computation of V, by the use of Plate I, is adequate.

Formulae (5) and (6) may be used either to obtain the value of E, when α, β, and γ are known; or to obtain the value of β, when α, γ and E are known. The total differentials of E_γ and E_α, for the first case, are as follows:

$$dE_\gamma=\frac{\sec E_\gamma}{\frac{1}{\alpha^2}-\frac{1}{\gamma^2}}\left[-\frac{1}{\alpha^3}\operatorname{cosec} E_\gamma(\beta^2-\sin^2 E_\gamma)d\alpha+\frac{\beta}{\alpha^2}\operatorname{cosec} E_\gamma d\beta-\frac{1}{\gamma^3}\sin E_\gamma d\gamma\right] \tag{22}$$

$$dE_\alpha=\frac{\sec E_\alpha}{\frac{1}{\alpha^2}-\frac{1}{\gamma^2}}\left[\frac{1}{\alpha^3}\sin E_\alpha d\alpha-\frac{\beta}{\gamma^2}\operatorname{cosec} E_\alpha d\beta+\frac{1}{\gamma^3}\operatorname{cosec} E_\alpha(\beta^2-\sin^2 E_\alpha)d\gamma\right] \tag{23}$$

For the mineral hornblende, take $\alpha=1.654$, $\beta=1.666$, $\gamma=1.670$ and E_α (computed)$=55°56'$. And assume as before observational errors of $\Delta\alpha=\Delta\gamma=.001$ and $\Delta\beta=-.001$. Then $\frac{dE_\alpha}{E_\alpha}=.379=38$ per cent, which is approximately the maximum possible error in E_α. This is still greater than the maximum error that results from the numerical computation of V; and is far greater than the error that results in the graphic computation of E from Plate II.

For the other use of formulae (5) and (6), in solving for β, when α, γ, and E are known, the total differentials are as follows:

$$d\beta=\frac{1}{\beta}\left[\frac{1}{\alpha}(\beta^2-\sin^2 E_\gamma)d\alpha+\frac{\alpha^2}{\gamma^3}\sin^2 E_\gamma d\gamma+\left(1-\frac{\alpha^2}{\gamma^2}\right)\sin E_\gamma\cos E_\gamma dE_\gamma\right] \tag{24}$$

$$d\beta=\frac{1}{\beta}\left[\frac{\gamma^2}{\alpha^3}(\sin^2 E_\alpha)d\alpha+\frac{1}{\gamma}(\beta^2-\sin^2 E_\alpha)d\gamma-\left(\frac{\gamma^2}{\alpha^2}-1\right)\sin E_\alpha\cos E_\alpha dE_\alpha\right] \tag{25}$$

For hornblende, take $\alpha=1.654$, $\gamma=1.670$, $E_\alpha=60.5°$ and β (computed) $=1.6656$. The maximum error will obtain when $\Delta\alpha$ and $\Delta\gamma$ are positive, and ΔE_α is negative or vice versa. Therefore we take $d\alpha=d\gamma=.001$ and $dE_\alpha=-0.5°$. We then have $\frac{d\beta}{\beta}=.00099=.1$ per cent, which is approximately the maximum possible error in β. This error is of the order of that which results from reading β directly from the chart. It therefore appears that chart II should be satisfactory for most minerals.

The total differential of V with regard to β and E, as derived from equation (3), is as follows:

$$dV=\frac{1}{\beta}\left[\frac{\cos E}{\cos V}dE-\tan V d\beta\right] \tag{26}$$

For the mineral hornblende, assume $\beta=1.666$, $E=60.5°$ and V (computed) $=31°29'$. The maximum error occurs when E and β have opposite signs. Hence taking $dE=0.5°$ and $d\beta=-.001$, we find that $\frac{dV}{V}=$six per cent. In Chart III, the minimum calibration of β is .02 and the minimum value to be obtained by interpolation is .01. Yet even with such readings of β, the value of V can be read closer than 6 per cent.

For reference, there are also given two other total differentials, derived from formula (4), (Mallard's formula), and formula (7). These are respectively as follows:

$$dE=\frac{1}{K}\left[\sec E dD-\tan E dK\right] \tag{27}$$

$$dV=\frac{1}{K\beta}\left[\sec V dD-\beta\tan V dK-K\tan V d\beta\right] \tag{28}$$

The total differentials above derived also lead to the consideration of another topic. Is it desirable, for example, to tabulate a value of V that is computed from α, β and γ, when the possible error in V may be as great as 28 per cent, as shown for hornblende? If the errors of observation are admitted to be as great as those assumed above, then, in the absence of other data, a more appropriate tabulation might be, $2V=60° \pm 16°$. The same considerations apply to the tabulation of V, as computed from equation (7), because there are initial errors in β, D and K. And if E is measured directly, by the use of a Fedorov stage, it is no more free of error than any other physical measurement; and E still has to be converted to V by means of formula (3), wherein one of the variables is β, which is admittedly subject to an observational error.

In view of these considerations, it may be desirable for the mineralogist or petrographer to determine, and possibly to publish, the probable errors of observation in the basic quantities, such as α, β, γ, D and K. Estimates of these primary errors could readily be made. Thereafter these initial errors could be inserted in the total differentials above given, to compute the approximate errors that may exist in the derived quantities, such as E and V. Or if desired, the possible errors in the derived quantities could be determined by substituting in the original formulae quantities like $\alpha+\Delta\alpha$, to replace α, where the increment in the independent variable represents the probable error in that variable. It is believed that tabulations of E and V would be more useful if their possible errors were thus indicated.

References

WRIGHT, FRED EUGENE, The methods of petrographic-microscopic research, plate 9. Washington, D. C. (**1911**).

WRIGHT, FRED EUGENE, Graphical methods in microscopical petrography: *Am. Jour. Sci.*, **36,** plates 5–7 (1913).

SMITH, HAROLD T. U., Simplified graphic method of determining approximate axial angle from refractive indices of biaxial minerals: *Am. Jour. Sci.*, **22,** 675–681 (1937).

LANE, J. H., JR., AND SMITH, H. T. U., Graphic method of determining optic sign and true axial angle from refractive indices of biaxial minerals: *Am. Jour. Sci.*, **23,** 457–460 (1938).

VON FEDOROV, E., Universalmethode und Feldspathstudien: *Zeits. Kryst.*, **26,** heft 3, plate IV, p. 260 (1896).

WRIGHT, FRED EUGENE, *Opus cit.*, 1911 and 1913.

BECKE, F., Klein'sche Lupe mit Mikrometer: *Tschermak's mineral. und petro. Mitt.*, **14,** heft 4, p. 377, (1904).

DE SOUZA-BRANDÃO, V., O novo microscopio da comissão do serviço geologico: *Communicações da comissão do serviço geologico de Portugal*, Tom. **5,** fasc. 1, 197–199, and plate 1 (1903).

SCHWARZMANN, MAX., Hilfsmittel, um die Ausrechnung der Mallard'schen Formel zu ersparen: *Neues Jahrb. Mineral., Geol. und Pal.*, Jahrgang 1896, **1,** plate II, 52–56 (1896).

LARSEN, ESPER S., AND BERMAN, HARRY, The Microscopic Determination of the Nonopaque Minerals, second edition: *U. S. Geol. Survey, Bull.* **848,** 95–213 (1934).