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INTERPOINT DISTANCES IN CYCLOTOMIC SETS*

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Abstract

Cyclotomic sets and their distance arrays were studied by Patterson. Here the distance arrays are treated as properties of self-images of cyclotomic sets. Although the sets are one-dimensional their self-images are two-dimensional. They have five kinds of plane symmetries, one for each of the five kinds of complementary pairs. A cyclotomic set is a simplification and a specialization of the ordinary Patterson map. Accordingly the distances in the cell of its self-image fall into three categories analogous to the three categories of vectors in the cell of a Patterson map. The distances in a cyclotomic set can be determined easily by inspection of its circular representation provided that the number of its points is quite small. In all cases the distances can be determined by making use of one of the important properties of the self-image array: all sets belonging to a specific cyclotomy can be constructed from any other such set by an exchange of one or more points with its complementary set. Such interchanges have applications in treating circular representations and especially in dealing with cyclotomic sets by the methods of image algebra.

Sommaire

Patterson étudia les ensembles cyclotomiques et leurs distributions de distances. Nous traitons ici ces distributions comme propriétés d'auto-images des ensembles cyclotomiques. Quoique les ensembles soient à une dimension, leurs auto-images sont bidimensionnelles. Elles possèdent cinq espèces de symétries planes, correspondant à cinq espèces de paires complémentaires. Un ensemble cyclotomique représente une simplification et spécialisation de la projection Patterson habituelle. Il s'ensuit que les distances à l'intérieur de la maille de l'auto-image de l'ensemble se répartissent en trois catégories analogues aux trois catégories de vecteurs que l'on distingue dans la maille de Patterson. Dans un ensemble cyclotomique, pourvu que le nombre de ses points soit petit, les distances se tirent à simple vue de sa représentation circulaire. Dans tous les cas, on détermine ces distances en se servant d'une des propriétés importantes de la distribution de l'auto-image: tout ensemble appartenant à une cyclotomie donnée peut se construire à partir de n'importe quel autre ensemble semblable par échange d'un ou de plusieurs points entre ce dernier et son complémentaire. Pareils échanges trouvent leur application dans le traitement des représentations circulaires et spécialement dans l'étude des ensembles cyclotomiques par l'algèbre des images.

(Traduit par la Rédaction)

*Dedicated to Professor J. D. H. Donnay on the occasion of his 75th birthday.

Introduction

Homometric sets

In determining the structure of bixbyite, Pauling & Shappell (1930) discovered two arrangements of metal atoms which scattered X-rays with the same intensities, yet were neither congruent nor enantiomorphic to each other. This implied that the two arrangements, though different, had the same set of interatomic vector distances. On learning of this property of some pairs of arrangements of points, Patterson (1939) referred to such pairs as *homometric.*

Later Patterson (1944) began a systematic study of periodic sets and demonstrated the existence of many homometric pairs among them. In this investigation he began by considering one-dimensional sets of points. He found it convenient to deal with a set whose translation was T as if it were wrapped around a circle whose circumference was T, as illustrated in Figure 1. This representation of a one-dimensional set may be called its *circular representation.* It has the advantage that, in the study of interpoint distances in the set, consideration is limited to those distances within one translation period. The interpoint distances then appear as arcs. To avoid the superposition of arcs in drawings, Patterson found it convenient to represent each arc by the chord it intercepts.

Patterson's initial search was among sets whose points were restricted to r of the N points which divided the translation T into N parts of equal length, t. Thus

$$t = T/N \qquad (N \text{ an integer}). \qquad (1)$$

The set of points which divides T into these N segments is called a *multiple lattice* (Buerger 1976). Patterson called the set of points which

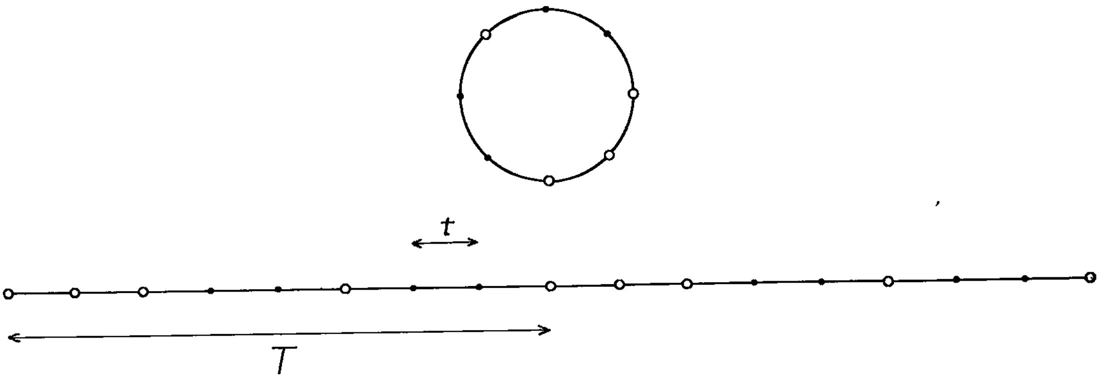

FIG. 1. The small circles on the horizontal line constitute a periodic one-dimensional set whose translation period is *T*. The set shown is specialized in that its points are restricted to some of the points of a lattice, called a *multiple lattice,* whose period *t* is a submultiple of *T*. In studying distances in a one-dimensional set it is convenient to wrap the set around a circle whose circumference is equal to *T,* as shown in the upper part of the illustration. This arrangement is called the *circular representation* of the set.

occupy *r* of the *N* points of the multiple lattice a *cyclotomic set.*

Tautoeikonic sets

It is convenient to have an adjective to characterize two or more sets which have the same interpoint distances without specifying whether the sets are congruent, enantiomorphic or homometric. For this more general relationship the adjective tautoeikonic ("having the same self-image") has been proposed (Buerger 1976).

DISTANCE ARRAYS AND SELF-IMAGES FOR CYCLOTOMIC SETS

The notion of images

The vector from a point *a* to a point *b* may be designated **ab**. (It cannot be represented by a pair of letters in boldface type since this is the standard designation of a dyadic.) The appearance of point *b* as seen from point *a* may be called the *image* of *b* from *a,* and may be designated by the label *ab*. If the set *A* consists of points *a, b, c, . . ,* the vectors between the points may be neatly displayed by writing the vectors in an ordered square array:

$$\begin{array}{ccccc} \overrightarrow{aa} & \overrightarrow{ab} & \overrightarrow{ac} & . & . \\ \overrightarrow{ba} & \overrightarrow{bb} & \overrightarrow{bc} & . & . \\ \overrightarrow{ca} & \overrightarrow{cb} & \overrightarrow{cc} & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{array} \qquad (2)$$

When all these vectors are transferred to the same origin, the set of images between the points is represented by a similar square array:

$$\begin{array}{ccccc} aa & ab & ac & . & . \\ ba & bb & bc & . & . \\ ca & cb & cc & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{array} \qquad (3)$$

By obvious extension of the notion of the image of one point from another point, array (3) is seen to represent the collection of images of the points of the set *A* from each other, including the self-image of each point. This collection is called the self-image of set *A,* and may be said to occur in *image space.*

The construction of ordered arrays

From diagrams (2) and (3) it is clear that each item in the distance array or self-image is the result of an interaction between points *a, b, c, . .* with each other. In constructing an array this can be aided by arranging a vertical margin with the characters *a, b, c, . .* somewhat to the left of the array to provide the first character of the line of pairs, and a horizontal margin with characters *a, b, c, . .* somewhat above the array to provide the second character to the column of pairs, thus:

$$\begin{array}{cccccc} & a & b & c & . & . \\ a & aa & ab & ac & . & . \\ b & ba & bb & bc & . & . \\ c & ca & cb & cc & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{array} \qquad (4)$$

This scheme is readily adapted to sketching diagrams of arrays in which some points of the multiple lattice are occupied by a subset while others are not. Only interactions between occupied points in a particular subset give rise to distance entries in the self-image.

Distances in cyclotomic sets

As cyclotomic sets are one-dimensional, all points of the set lie on a single straight line, so the vectors of (2) degenerate into positive and negative distances in image space. Furthermore the points are located at some of the points of a one-dimensional multiple lattice, as in Figure 2. Let the set of all such possible point locations be

$$M = \gamma_1, \gamma_2, \gamma_3, \ldots \gamma_N\,, \qquad (5)$$

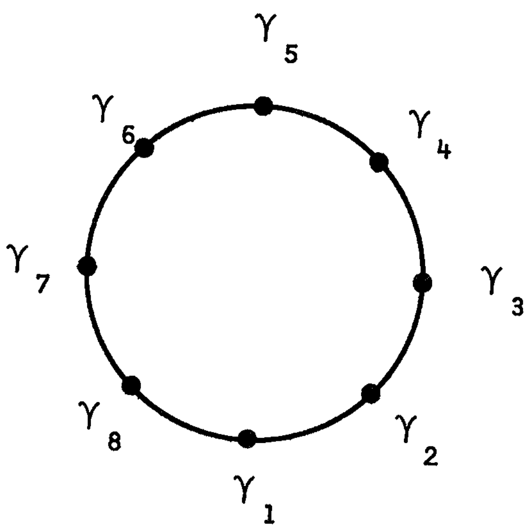

FIG. 2. The circular representation of the points of a one-dimensional multiple lattice with $T/t = 8$. The set consists of the points in equation (5) in ordered sequence 1, 2, 3, . . N. Here the first point γ_1 is placed at an origin at the bottom of the circle.

where the subscripts indicate that the points are enumerated in an ordered sequence from the origin. The self-image of this set is

$$MM = \begin{array}{cccccc} \gamma_1\gamma_1 & \gamma_1\gamma_2 & \gamma_1\gamma_3 & . & . & \gamma_1\gamma_N \\ \gamma_2\gamma_1 & \gamma_2\gamma_2 & \gamma_2\gamma_3 & . & . & \gamma_2\gamma_N \\ \gamma_3\gamma_1 & \gamma_3\gamma_2 & \gamma_3\gamma_3 & . & . & \gamma_3\gamma_N \\ . & . & . & . & . & . \\ \gamma_N\gamma_1 & \gamma_N\gamma_2 & \gamma_N\gamma_3 & . & . & \gamma_N\gamma_N \end{array} \qquad (6)$$

All these points are at distances from the origin which are multiples of the multiple-cell translation $t = \gamma_i\gamma_{i+t}$. Because of this simplicity, the points of (6) need not necessarily be labeled, but may often be represented simply by a square array of dots.

For a particular cyclotomic set A whose points are located at r of the N points of the set M, the distances in (2) are, in general, different for different arrangements of the r occupied points. Thus the self-image AA of a set A is, in general, different from the self-image BB of a different set B. Examples of two simple but different sets and their distance arrays are shown in Figure 3, where the points of the distance arrays are indicated by dots whereas the points occupied in sets A and B are indicated by small circles.

Some properties of distance arrays

When the points in M are ordered, then the distance array for the self-image in (6) has some useful properties. Any specific row contains the distances from a specific point to all points in M (including the null distance from the specific point to itself), whereas the corresponding column contains the distances from all points in M to that specific point. Corresponding row and column cross at the distance from the specific point to itself. Any corresponding row and column contain, point-by-point, distances that have the same magnitudes but opposite directions. Thus, the sequence of distances in a column are the negative equivalents of the sequence of distances in the corresponding row. The main diagonal of the distance

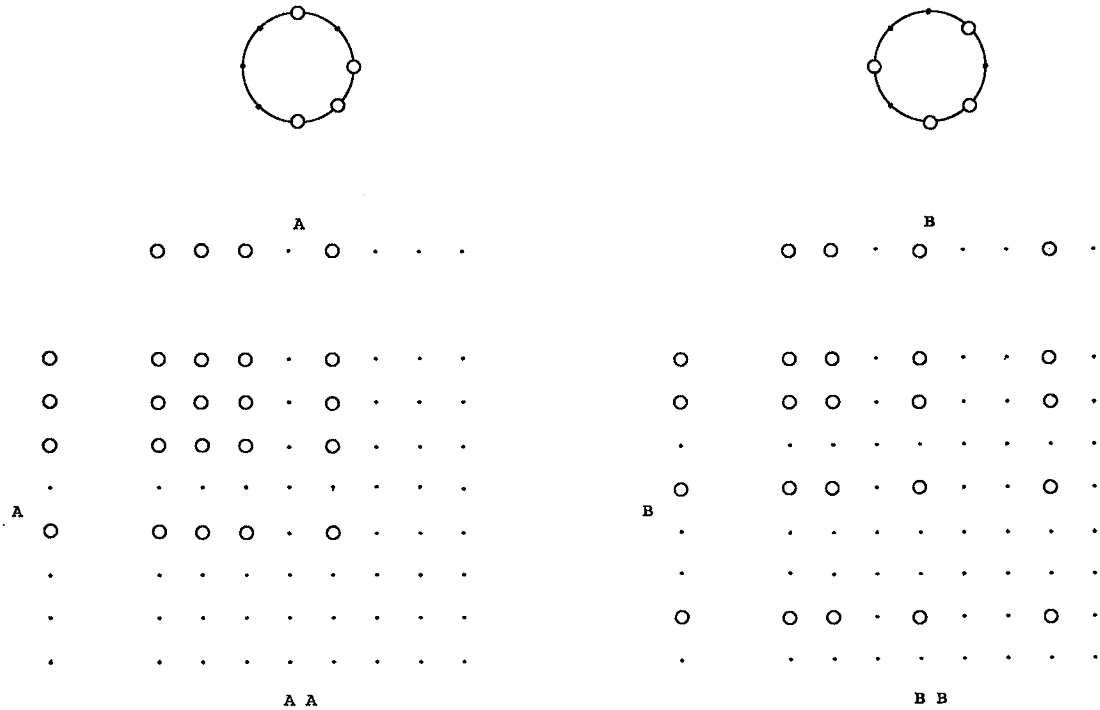

FIG. 3. The upper part of the illustration shows the circular representations of two different cyclotomic sets A and B, both with $N = 8$, $r = 4$. The lower part of the illustration shows the self-image AA of set A and the self-image BB of set B. From these diagrams the interpoint distances in sets A and B can be determined.

array, therefore, behaves like a line of mirror symmetry between the magnitudes in the columns and rows.

The main diagonal contains distances between points with labels like $\gamma_1\gamma_1$, $\gamma_2\gamma_2$, $\gamma_3\gamma_3$, . ., which are the self-images of γ_1, γ_2, γ_3, . .; these correspond to positive and negative distances which are identical and so must be zero. This feature of ordered arrays corresponds to the origin peak of the Patterson function. But for cyclotomic sets the array has another property not shared with the general Patterson function. Because of the restriction of points of a cyclotomic set to some of the locations of the points of the multiple lattice M whose primitive translation is t, any line parallel to the main diagonal but separated from it by a spacing of p lines can contain only distances between points like $\gamma_n\gamma_{n+p}$ for a line above the main diagonal, but can contain only distances between points like $\gamma_n\gamma_{n-p}$ for a line below the main diagonal, as illustrated in Figure 4. This property permits determining the collection of distances in a cyclotomic set by a quick inspection of the occupied points in the several lines parallel to the main diagonal. The collection of these

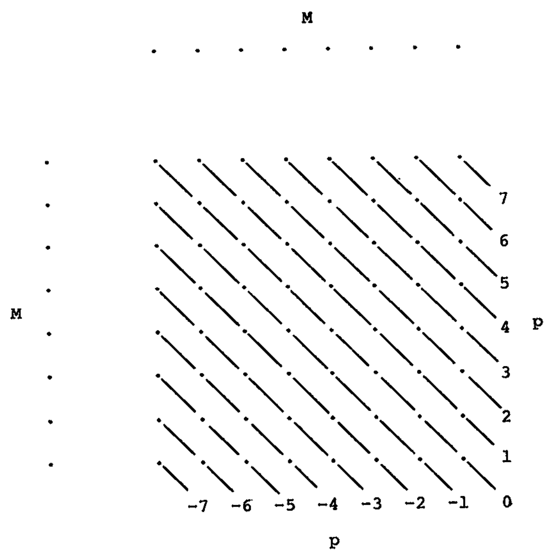

FIG. 4. A diagram of the self-image MM of the points of a multiple lattice M within the range T. All points which lie on a particular line parallel to the main diagonal of the self-image array correspond to the same interpoint distance p.

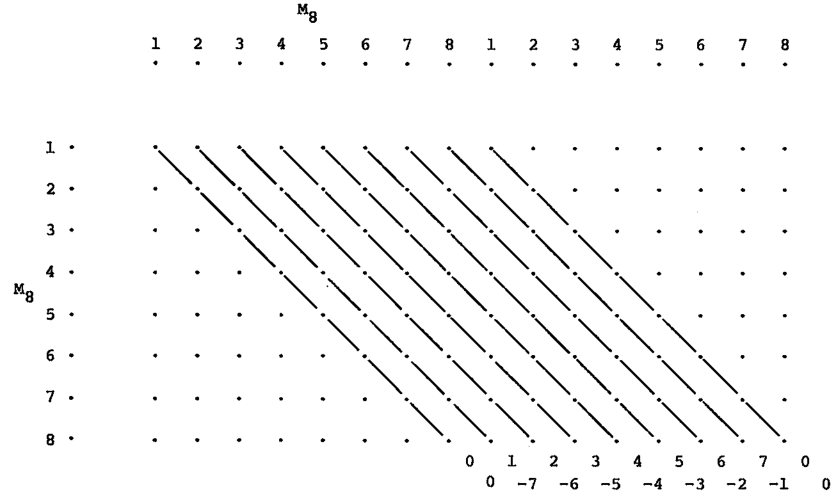

Fig. 5i

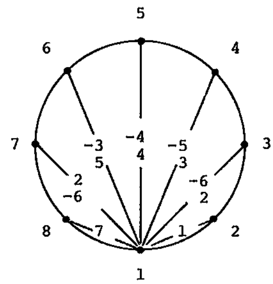

Fig. 5ii

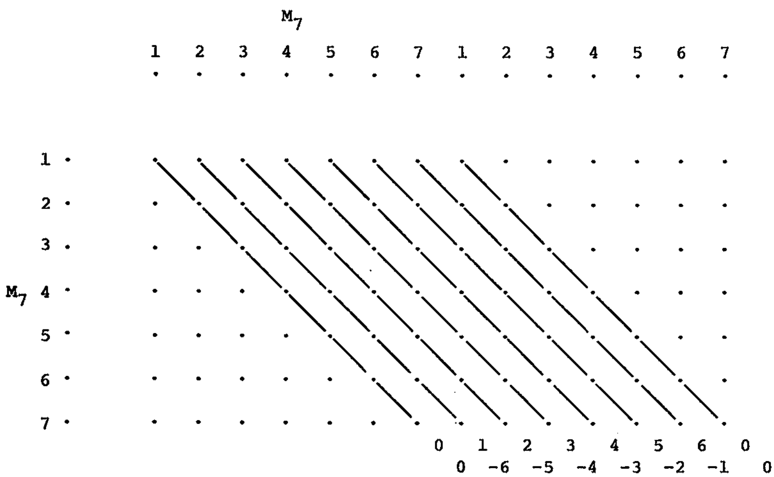

Fig. 6i

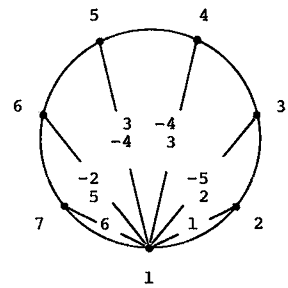

Fig. 6ii

FIGS. 5 and 6. Consideration of adjacent cells of the self-image in Figure 4 make it evident that a line parallel to the main diagonal may refer to both positive and negative distances. The distribution of these distances is different depending on whether N is even, shown in Figure 5, or odd, shown in Figure 6.

distances is a characteristic of the self-image of the cyclotomic set.

From the appearance of Figure 4 the impression might be gained that, in the distance array of M, the number of points in a line parallel to the main diagonal decreases with p. But when these lines are drawn for adjacent cells, as in Figure 5i, it becomes evident that p, as referred to one cell, is the same as $N-p$ referred to the adjacent cell; this is a consequence of the modular nature of translation repetitions. Thus, for set M, lines drawn parallel to the main diagonal all contain the same number of points, namely N, and each such line of points corresponds to N distances, each of p times the unit translation t.

The relation of numbered points to the distances from one of them is displayed on a circular diagram in Figure 5ii. The distribution of distances is somewhat different for N even, illustrated in Figure 5ii, and for N odd, shown in Figure 6ii. For N odd, all chords occur in pairs that are symmetrical about a diameter, but for N even, one singular chord lies along the symmetry diameter and corresponds to the distance $+N/2$ and $-N/2$.

Some of this information can be expressed as follows:

Lemma 1: In the distance array of a multiple-lattice set, the nth row contains the distances from the point which occurs at a distance of $nt = nT/N$ from the origin of the original set to all points of the set, and the nth column contains the distances from all points of the set to the point at a distance of $nt = nT/N$ from the origin of the original set. Thus successive rows (and corresponding columns) contain the same distances in cyclical permutation.

Theorem 1: A multiple lattice M, described by N ordered points with translation interval t, has a self-image MM consisting of N points arranged in a square array. These points are also aligned along N equally spaced lines parallel to the main diagonal. Any line located p spacings from the main diagonal contains N points, each of which occurs at the same distance pt from the origin of image space, where p runs from 0 to N–1.

The self-image array thus provides a useful device for permitting the counting of interpoint distances by inspection.

Unoccupied points

In cyclotomic sets, some points of the multiple-lattice set are, in general, occupied; others are not. If a point is occupied, there are interpoint distances between that point and other occupied points of the set. If any point of the set is unoccupied there are no meaningful distances within the set, between that point and any other points of the set. This is illustrated in Figure 7, where set A lacks the 4th point in M. This situation leads to

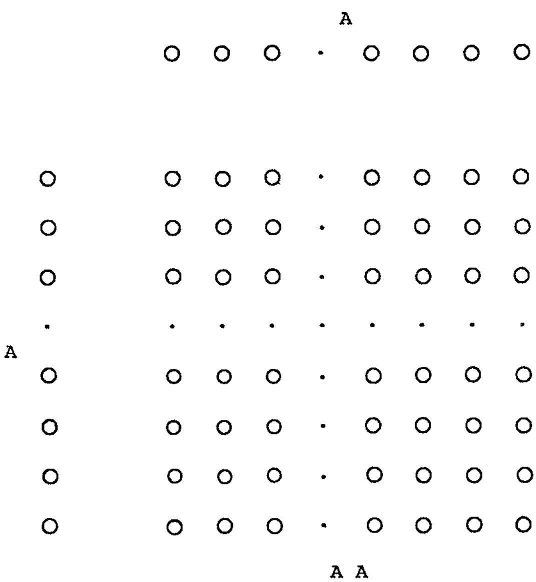

FIG. 7. Demonstration that if a node of the multiple lattice is unoccupied by a point in set A, the corresponding row and column of the self-image AA are empty.

Lemma 2: If any point of a multiple-lattice set is unoccupied the entire row and entire column corresponding to the unoccupied point are absent in the self-image array.

Lemma 3: If a row or column of a distance array contains the points of one subset, it cannot contain the points of another subset.

Lemma 4: The self-images of subsets are distinct. No subset shares a row or column with any other subset.

Complementary sets

If a cyclotomic set A occupies r of the N points of the multiple-lattice set, a set consisting of the remaining $N - r$ points is called the set complementary to A. The complementary set is labeled A', following a practice by Patterson (1944). Complementary sets play an important role in the study of tautoeikonic sets.

A pair of complementary sets constitutes a special case of two subsets for, whereas two subsets do not necessarily exhaust the points of the multiple lattice, the sum of the points of any pair of complementary sets is equal to all the N points of the multiple lattice, giving the important relation

$$A + A' = B + B' = M. \tag{7}$$

More specific statements for Lemmas 2 and 3 as applied to complementary sets are:

Theorem 2: If a row or column of the self-image array contains the points of a set, it cannot contain points of the complementary set.

Theorem 3: Every row or column of the self-image array contains either the r points of a set or the $N - r$ points of its complementary set.

In most applications the set and its complement are distinct so that distances between the two sets are unimportant. The number of points occupied in the distance array by the set and its complement depends on both N and r:

Theorem 4: The number of points in the self-images of a set and its complement is $N^2 - 2r(N - r)$. This is a minimum when $r = \frac{1}{2}N$,

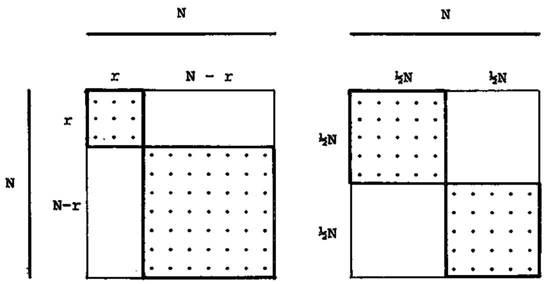

FIG. 8. Demonstration that the sum of the points in the self-images of a set and its complement is $N^2 - 2r(N - r)$. The minimum sum occurs when $r = \frac{1}{2}N$.

when the number of points in the two self-images is $\frac{1}{2}N^2$.

This is illustrated in Figure 8.

A SYMBOLISM FOR CYCLOTOMIC SETS

When discussing cyclotomic sets it is desirable to have a symbolism for the purpose of distinguishing one set from another. A simple numerical basis for a suitable symbolism (Buerger 1977) follows. In general, a cyclotomic set consists of several sequences, each consisting of several points occupied without a gap, then each such sequence separated from the next by a gap consisting of unoccupied points. Each uninterrupted sequence can be represented by a numeral corresponding to the number of occupied points, and each gap can be represented by a numeral corresponding to the number of its unoccupied points. The symbol of the set is then the alternating sequence of these two kinds of numerals. To distinguish occupied points from unoccupied points, the former are written at an elevated level and the latter at a lower level. An example is $^5{}_2{}^1{}_3{}^2{}_3$. It is easy to see that the number of upper numerals is equal to the number of lower numerals, and that the sum of all the numbers is N.

SYMMETRIES OF SELF-IMAGE ARRAYS

Modular coordinate systems

Because the cells of a periodic structure are repeated by translation, the points in the cells can be referred to a modular coordinate system. For cyclotomic sets this modulus is N. If N is 8, a point a at $x = 5$ can be located equally well by $x = -3$ because $-3 \equiv 5 \bmod 8$, and another point b at $x = 7$ can be located equally well at $x = -1$ because $-1 \equiv 7 \bmod 8$. In these statements it is assumed that an origin has been chosen at one of the points of the multiple lattice M which has been assigned the coordinate $x = 0$.

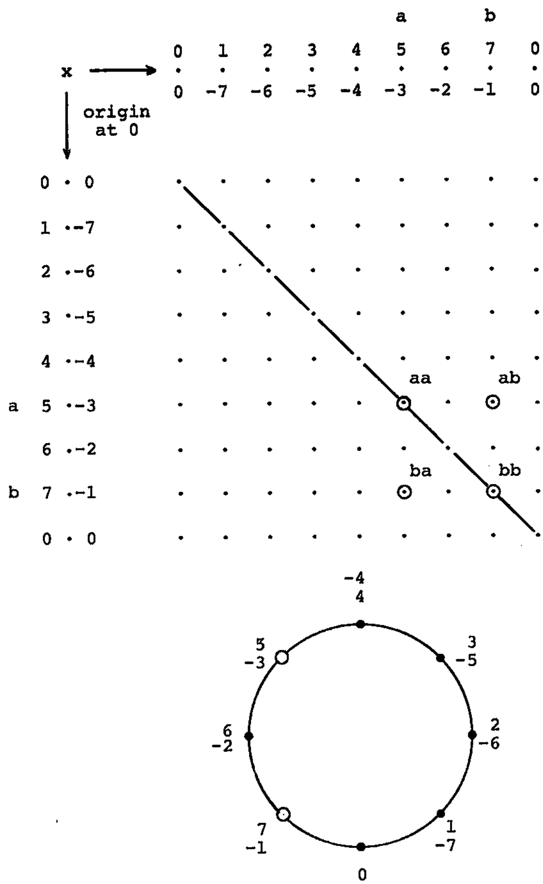

FIG. 9. The relation of points in the self-image of a set which consists of the two points a and b. If the coordinates of the points of the self-image are important the origin of the set is designated zero.

The set so defined is shown along the borders of Figure 9. The self-image of the set is: $(a + b)\ (a + b) = aa + ab + ba + bb$. It is evident from Figure 9 that the coordinates of these points in the self-image array are symmetrical with respect to a mirror line running along the main diagonal.

The symmetries of individual cyclotomic sets

An individual cyclotomic set, that is, a single set occupying r or the N points of a multiple lattice, can fall into one of only two classes with respect to symmetry: asymmetric or symmetric (Buerger 1977). If the set is asymmetric, its self-image array nevertheless has a certain definite symmetry. An example for

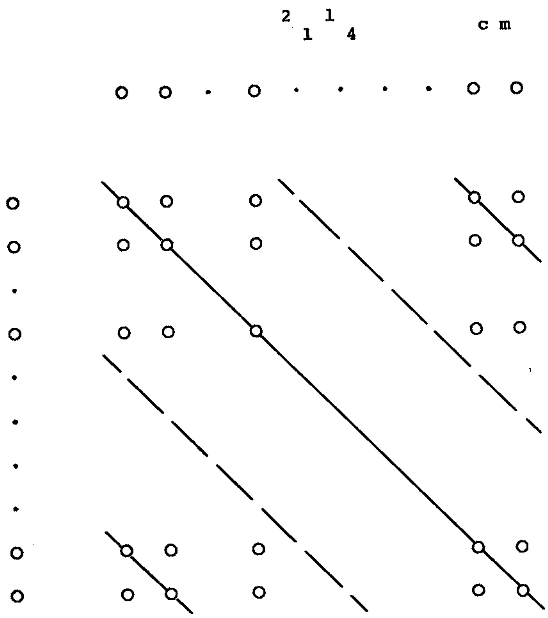

FIG. 10. Demonstration that a set without symmetry has a self-image whose symmetry is *cm*. This symmetry is also a proper subgroup of a symmetric set. The full line is a line of mirror symmetry; the broken line is a line of glide symmetry whose translation component is $\frac{1}{2}N\sqrt{2}$.

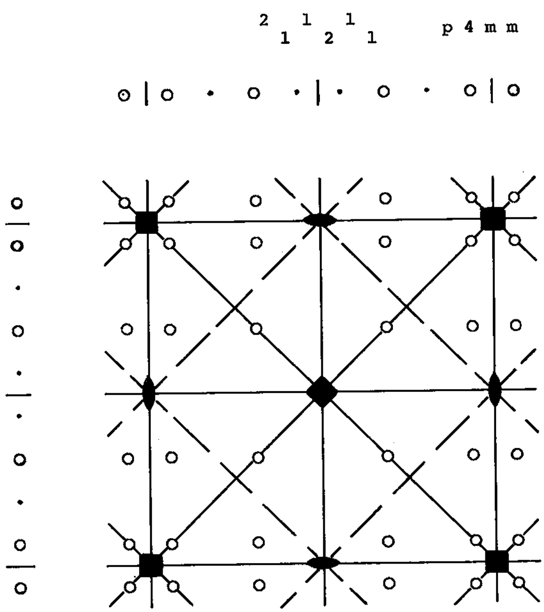

FIG. 11. Demonstration that a symmetric set has a self-image array whose symmetry is *p2mm*.

$N = 8$ is shown in Figure 10. It can be seen that the main diagonal relates points as a mirror line; the parallel line half-way between main diagonals relates the points of the array as a glide line whose translation component is $(N/2)\ \sqrt{2}$. The resulting symmetry of the self-image array is that of plane group *cm*.

If the set is symmetric, the symmetry elements can be represented by inversion centres, or by mirror lines normal to the translation direction. In this case the symmetry of the self-image of the set is that of the plane group *p4mm*. An example is shown in Figure 11 for $N = 8$. The new rotational symmetry elements can be regarded as due to the projections of the symmetry elements of the sets in the two margins into the interior of the self-image array. This projection occurs because, if the set in the upper margin has a symmetry element at a specific location, the image of the set from any point in the left margin (and vice versa) has a corresponding symmetry element at the corresponding location. Thus any mirror line in the sets along the margins extends through the body of the self-image array. They cross the mirror lines along the main diagonals and the glide lines half-way between them at angles of 45°. At their intersections with the mirror lines they generate 4–fold rotors, whereas at their intersections with the glide lines they generate 2-fold rotors.

The symmetries of complementary pairs

It has been shown (Buerger 1977) that there are five kinds of complementary pairs: asymmetric homometric, symmetric homometric, enantiomorphic, asymmetric congruent and symmetric congruent. Each of these varieties has a characteristic self-image array with a characteristic symmetry. The last three involve antisymmetry, and so conform to certain of the black-white symmetries. As the usual geometrical representation of symmetry elements in diagrams of antisymmetries requires the use of two colors, these symmetry elements are omitted in illustrations accompanying the discussions of such symmetries given below.

Homometric pairs. The least specialized of the tautoeikonic complementary pairs are the two homometric varieties. In neither of these are the set and its complement related to one another by a coincidence operation. Accordingly, there are no other symmetry relations than those shown in Figures 10 and 11. Asymmetric homometric pairs therefore have self-images whose symmetries conform to the *cm* of Figure 10, and symmetric homometric pairs have self-images whose symmetries conform to the *p2mm* of Figure 11.

Enantiomorphic pairs. Each member of an enantiomorphic pair must be asymmetric, so the symmetry of the contribution of each set is that of Figure 10. The two sets, however, are related to one another by antireflection mirrors normal to the translation. The same antisymmetry must also relate the corresponding points on the main diagonal. This correspondence locates the antisymmetry elements which relate the independent self-images of the two members of the complementary pair. The resulting symmetry of the entire self-image of the pair is $c2'mm'$. An example for $N = 8$ is shown in Figure 12 for $^{3}{}_{3}{}^{1}{}_{1}$.

Asymmetric congruent pairs. Each member of the pair is asymmetric, so the contribution of each set to the symmetry of the self-image is that of Figure 10. The two sets, however, are related to one another by a submultiple antitranslation t'. This coincidence operation also occurs mapped along the main diagonal. This location is mentioned merely to reveal the direction of the antitranslation in the self-image array, for indeed it occurs throughout the entire self-image array; its length is half the period of the main diagonal and its direction is that of

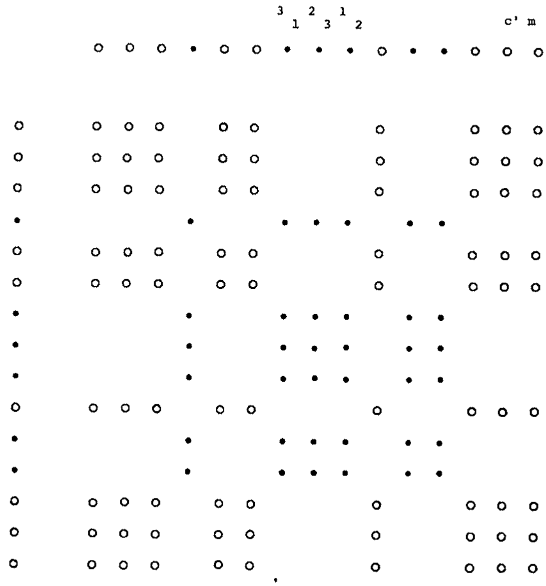

FIG. 13. The symmetry of the self-images of a complementary pair of cyclotomic sets that are asymmetric congruent. The symmetry elements and translations of the resulting $c'm$ are omitted for clarity.

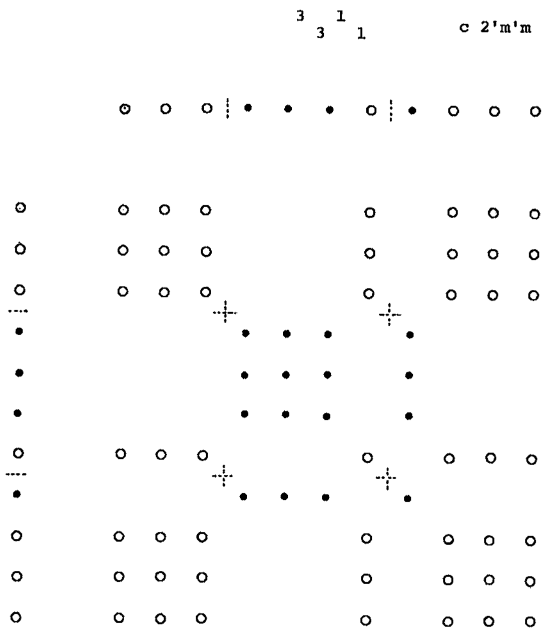

FIG. 12. The symmetry of the self-images of a complementary pair of cyclotomic sets that are enantiomorphic. The sets are related by two antireflection mirrors per translation; these are indicated by short dotted lines orthogonal to the translation. The projections of these into the self-image array produces the four dotted crosses. These lead to the symmetry $c2'm'm$ for the self-image array. The symmetry elements are omitted for clarity.

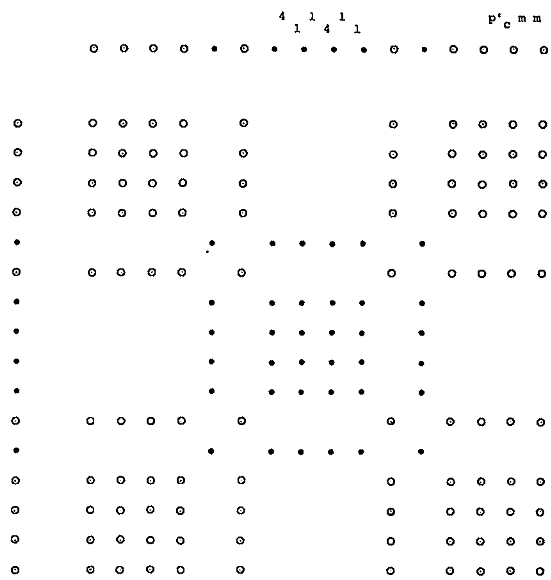

FIG. 14. The symmetry of the self-images of a complementary pair of cyclotomic sets that are symmetric congruent. The symmetry elements and translations of the resulting $p'_{c}mm$ are omitted for clarity.

TABLE 1. CORRELATION OF THE FEATURES OF THE CELL OF THE SELF-IMAGE ARRAY WITH THE FEATURES OF THE CELL OF THE PATTERSON MAP

Number of peaks in the cell of the Patterson map		Number of distances in the cell of the self-image array	
Total number:	n^2	Total number:	r^2
Peaks coalescing at the origin:	n	Distances of magnitude zero:	r
Peaks at locations x y z:	$\frac{1}{2}(n^2 - n)$	Obverse distances, 1t to ½Nt:	$\frac{1}{2}(r^2 - r)$
Peaks at locations $\bar{x}\ \bar{y}\ \bar{z}$:	$\frac{1}{2}(n^2 - n)$	Reverse distances, ½Nt to (N-1)t:	$\frac{1}{2}(r^2 - r)$

the main diagonal. A second antitranslation of equal magnitude occurs at right angles to the first one. These antitranslations add to the mirror along the main diagonal a coincident antireflection glide line and to the glide line between main diagonals, an antireflection mirror coincident with the original glide line. The resulting symmetry is *c'm*. An example for $N = 12$ is shown in Figure 13 for $^{3}{}_{1}{}^{2}{}_{3}{}^{1}{}_{2}$.

Symmetric congruent pairs. Each member of the pair is symmetric, so the contribution of each set to the symmetry of the self-image is that of Figure 11. As in the case of the asymmetric congruent pair, the two sets are related by a submultiple antitranslation t'. When this is mapped along the main diagonal, as above, it is evident that the self-images of the two independent sets of the pair are related by two orthogonal antitranslations whose lengths are half the diagonal of the self-image cell. These antitranslations augment the symmetries *p4mm* of the independent parts of the self-image to the antisymmetry $p_c'4mm$. An example for $N = 12$ is shown in Figure 14 for $^{4}{}_{1}{}^{1}{}_{4}{}^{1}{}_{1}$.

Distance Arrays for Complementary Sets

Obverse and reverse distances

In determining distances from a self-image array, it is sometimes useful to distinguish between two kinds of distances. It was pointed out that the main diagonal separates positive distances on its upper right from negative distances on its lower left. It was also noted that the same relation occurs in adjacent cells, as a consequence of which the positive and negative distances cover the same range between adjacent cells in opposite directions. The distances in the region from the main diagonal to the nearest glide line on its upper right may be termed *obverse* distances, whereas those from the main diagonal to the nearest glide line on its lower left may be termed *reverse* distances.

In discussing Figures 5 and 6 it was shown that there is a difference in the arrangement of distances for a multiple-lattice set M depending on whether N is odd or even. A similar difference occurs for the distances of a set A. When r is odd, the distances in the array are aligned along an *even* number of lines parallel to the main diagonal; these occur in pairs of lines, one with obverse, the other with reverse distances, symmetrical about the glide line. When r is even, the number of lines is odd, so the middle line coincides with the glide line. In this case both obverse and reverse distances occur on this particular line. Attention to this feature will be given again later.

Patterson maps and self-image arrays

The self-image array AA is essentially a simplification and a specialization of the familiar Patterson map. It is a simplification because cyclotomic sets are one-dimensional; it is a specialization because the points of the set are limited to a certain number of the N points of the multiple lattice. Accordingly the distribution of points in the cell of the Patterson map and of the points of the cell of a self-image array fall into similar categories, shown in Table 1. This brings out the fact that, for a set of r points, the number of points in each of the three categories of the cell of the array is always r, $\frac{1}{2}(r^2 - r)$ and $\frac{1}{2}(r^2 - r)$ regardless of the pattern of distribution of the r points over the N points of the multiple lattice. The only variation between sets having the same values of r and N is the distribution of distances over the possible values $t, 2t, 3t, \ldots (N-1)t$.

Determination of interpoint distances

The distances in a cyclotomic set can be determined in several ways. In sets with small values of N they can be easily determined from the circular representation by noting the lengths of the arcs intercepted by the collection of chords drawn between all pairs of points of the set. An example of these is given in Table 2, in which the distances between points are derived from the sets of Figure 15. This procedure becomes tedious for sets with any but small values of N because the diagrams are cluttered with criss-crossing chords.

The distances in any set can also be deter-

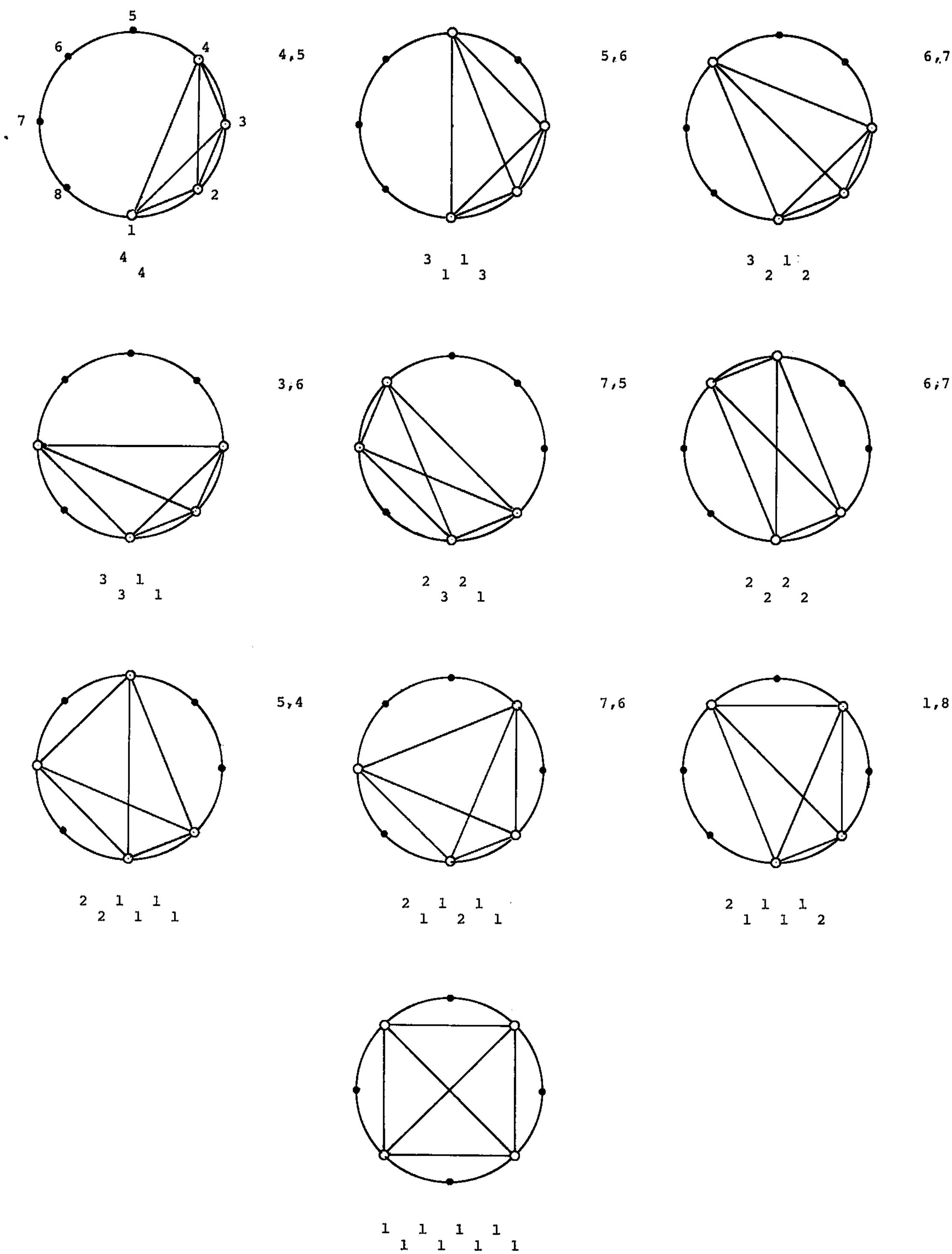

FIG. 15. The circular representations of the 10 distinct cyclotomic sets with $N = 8$, $r = 4$. Below each representation is given the symbol of the set. The pair of numbers between sets indicates the interchange of points (as numbered in the upper left diagram) to transform a set into the neighboring set.

TABLE 2. THE TEN CYCLOTOMIC SETS HAVING $N = 8$, $r = 4$; THEIR INTERRELATIONS AND THEIR INTERPOINT DISTANCES

Complementary pairs A	A'	Symmetry	Relation of A to A' and reciprocity	Number of distances of amount 0t	1t	2t	3t	4t
4 4	4 4	S	congruent	4	3	2	1	
3 1 1 3	3 1 3 1		enantiomorphic ┐	4	2	2	1	1
3 1 2 2	2 2 3 1	S	┌ homometric	4	2	1	2	1
3 1 3 1	3 1 1 3		enantiomorphic ┘	4	2	2	1	1
2 2 3 1	3 1 2 2	S	└ homometric	4	2	1	2	1
2 2 2 2	2 2 2 2	S	congruent	4	2		2	2
2 1 1 2 1 1	2 1 1 1 1 2		enantiomorphic ┐	4	1	2	2	1
2 1 1 1 2 1	2 1 1 1 2 1	S	congruent	4	1	2	3	
2 1 1 1 1 2	2 1 1 2 1 1		enantiomorphic ┘	4	1	2	2	1
1 1 1 1 1 1 1 1	1 1 1 1 1 1 1 1	S	congruent	4		4		2

mined by inspection of the self-image array of the set. The background for this method is supplied by *Theorem 1*. The method is especially useful in dealing with pairs of complementary sets. In the context of such sets the distances in a set A and the distances in its complementary set A' are independent of each other, and the self-images of these complementary pairs are ordinarily considered as independent features.

The application of self-image arrays to the determination of distances is illustrated by the

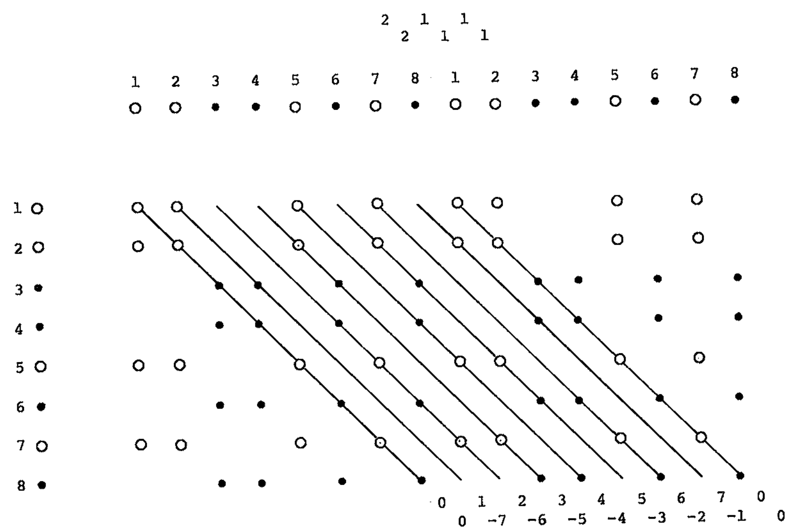

FIG. 16. An example of how all distances within each of two complementary cyclotomic sets can be determined by making use of the relation demonstrated in Figure 5. The distances derived from this diagram and from the other distinct sets with $N = 8$, $r = 4$ are listed in Table 3.

TABLE 3. NUMBERS OF DISTANCES, IN UNITS OF t, IN THE 10 CYCLOTOMIC SETS HAVING $N = 8$, $r = 4$

From polygon vertex number:	Symbol of set				
	4 4	3 1 1 3	3 1 2 2	3 1 3 1	2 2 3 1
1	0 1 2 3	0 1 2 . 4 . . .	0 1 2 . . 5 . .	0 1 2 . . . 6 .	0 1 . . . 5 6 .
2	0 1 2 7	0 1 . 3 . . . 7	0 1 . . 4 . . 7	0 1 . . . 5 . 7	0 . . . 4 5 . 7
3	0 1 . . . . 6 7	0 . 2 . . . 6 7	0 . . 3 . . 6 7	0 . . . 4 . 6 7	0 1 . 3 4 . . .
4	0 . . . . 5 6 7	0 . . . 4 5 6 .	0 . . 3 4 5 . .	0 . 2 3 4 . . .	0 . 2 3 . . . 7
Number†	4 3 2 1 . 1 2 3	4 2 2 1 2 1 2 2	4 2 1 2 2 2 1 2	4 2 2 1 2 1 2 2	4 2 1 2 2 2 1 2
Number§	4 3 2 1 .	4 2 2 1 1	4 2 1 2 1	4 2 2 1 1	4 2 1 2 1

From polygon vertex number:	Symbol of set				
	2 2 2 2	2 1 1 2 1 1	2 1 1 1 2 1	2 1 1 1 1 2	1 1 1 1 1 1 1 1
1	0 1 . . 4 5 . .	0 1 . . 4 . 6 .	0 1 . 3 . . 6 .	0 1 . 3 . 5 . .	0 . 2 . 4 . 6 .
2	0 . . 3 4 . . 7	0 . . 3 . 5 . 7	0 . 2 . . 5 . 7	0 . 2 . 4 . . 7	0 . 2 . 4 . 6 .
3	0 1 . . 4 5 . .	0 . 2 . 4 5 . .	0 . . 3 . 5 6 .	0 . 2 . . 5 6 .	0 . 2 . 4 . 6 .
4	0 . . 3 4 . . 7	0 . 2 3 . . 6 .	0 . 2 3 . 5 . .	0 . . 3 4 . 6 .	0 . 2 . 4 . 6 .
Number†	4 2 . 2 4 2 . 2	4 1 2 2 2 2 2 1	4 1 2 3 . 3 2 1	4 1 2 2 2 2 2 1	4 . 4 . 4 . 4 .
Number§	4 2 . 2 2	4 1 2 2 1	4 1 2 3 .	4 1 2 2 1	4 . 4 . 2

† Number of distances from all four polygon vertices.

§ Number of absolute magnitudes: number of distances from all four polygon vertices for 0t, 1t, 2t, 3t, but half the number for 4t.

self-image of one of the 10 sets of Figure 15; this is shown in Figure 16. The distances determined from similar arrays of all the 10 sets of the $N = 8$, $r = 4$ cyclotomy are shown in Table 3, where the sequence of sets is the same as in Figure 15. The theory of the occurrence of these distances in self-image arrays calls for a little explanation.

From *Lemma 1* and *Theorem 1* it is clear that each horizontal line of the self-image array is occupied by a series of distances from a particular point to the r points of the set. The total number of distances in the set is the sum of these for each of the r lines, the total amounting to r^2 distances. These include zero distances, obverse distances and reverse distances as listed in Table 1. The specific distances from each of the r points to the r points of the set, in terms of $0t$, $1t$, $2t$, . . $(N - 1)t$ are given in the interiors of the 10 blocks of numbers in Table 3, one block for each cyclotomic set. Immediately below the block is given the sum of the number of distances in each category, as a line of 8 numbers (where a dot implies that no distance was observed in that category). Below this line of 8 numbers is a line of 5 numbers which give the absolute distances (including the four zero distances). For the first four categories these absolute distances are the same as the obverse distances above them, but the number for category $\frac{1}{2}Nt = 4t$ of the line above the number must be divided by two because r is even, as explained in the last section.

Transformations of Order

Interchanges of points

For a specific pair of values of N and r (provided $N \geqq 4$ and $r \neq 1$) there exists a number of distinct sets. Starting with any one of these, the others can be derived from it as follows:

Lemma 5: Any possible cyclotomic set A and its complement A' can be derived from any other set of the same cyclotomy N,r by interchanging one or more points of A with the same number of points of A'. If the transformation to the new set requires several interchanges, the result is independent of the order in which the interchanges are made, or whether they are made together or separately.

This lemma is valid because it depends on permutations of some of the N points of the multiple lattice in which the interchange occurs only between two independent fractions of N. (An interchange of points contained in A with each other, or an interchange of points contained in A' with each other, obviously leaves both A and A' unchanged.)

Application to circular representations

Such interchanges provide a device for studying changes in distances between sets of the same N,r cyclotomy. A simple example is afforded by the sets having $N = 8$, $r = 4$. In this cyclotomy there are 10 distinct sets A, shown in circular representation in Figure 15. In these diagrams small empty circles are points of the set A whereas large black dots are points of the complementary set A' In the upper-left diagram the points of the multiple lattice M are assigned numbers from 1 through 8. A single interchange involves interchanging the points at a pair of numbers. The arrangement of the diagrams is such that, to transform a particular set to one on its left or right, a single interchange is made of the pair of points whose numbers are noted between the original and transformed sets. Transformations can be performed toward the right or the left. Thus the entire collection of 10 sets can be constructed step-by-step from any one of them by a sequence of single interchanges. Alternatively, any of the first four sets can also be transformed into one another directly by a single interchange, and any of the 10 sets of the cyclotomy can be transformed into one another by no more than two interchanges.

Some of the properties of the sets of Figure 15 are given in Table 2. The first (double) column gives the symbols for A and A'; the second column indicates by S which of the sets are symmetric; the third column expresses the relation between the complementary sets and also ties together the reciprocal pairs (Buerger 1977). The remaining five columns give the absolute distances for each set A. For this cyclotomy, N is small enough so that the chords (seen in Fig. 15) do not make a serious clutter.

Application to image algebra

The proof of Patterson's Theorem (ii) with the aid of image algebra (Buerger 1976) made use of the transformations of order to construct other sets of the same cyclotomy. The general strategy of this application was that it is possible to begin the transformation with a set A and its complement A' which are congruent and which, therefore, have the same distances. If an arbitrary point a in A is interchanged with an arbitrary point b in A', then A and A' are transformed into new sets B and B' according to *Lemma 5*. The relation of B and B' to A and A' are expressed in image algebra as

$$B = A - a + b \tag{8}$$

$$B' = A' + a - b, \tag{9}$$

from which the self-images of these sets are

$$BB = (A - a + b)(A - a + b) \tag{10}$$

$$B'B' = (A' + a - b)(A' + a - b). \tag{11}$$

If B and B' are tautoeikonic, that is, have the same distances, then the difference between (10) and (11) must be zero. By manipulating (10) and (11) by the rules of image algebra and applying a simple but powerful theorem (Buerger 1976) it can be demonstrated that this is true, so that B and B' are tautoeikonic.

This procedure can be continued by interchanging an arbitrary point c in B with an arbitrary point d in B' to produce a new pair of complementary sets C and C'. The image-algebra expression of this interchange is

$$C = B - c + d \tag{12}$$

$$C' = B' + c - d. \tag{13}$$

As (12) and (13) have the same form as (8) and (9) their self-images are similar to (10) and (11). These self-images can be reduced by image algebra with the result that $CC = C'C'$, so that C and C' are also tautoeikonic. This procedure can obviously be continued until every pair of complementary sets in the cyclotomy are shown to be tautoeikonic.

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