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INDIRECT DETERMINATION OF THE PRINCIPAL REFRACTIVE INDICES OF BIAXIAL CRYSTALS

RIASSUNTO. — Ad ogni intersezione del piano di una sezione qualsiasi con uno dei piani principali della sua indicatrice corrisponde un raggio che è comune sia ad una delle ellissi sezioni principale dell'indicatrice, sia all'ellisse corrispondente al piano della sezione. Se di questa si conoscono i due indici di rifrazione e l'orientazione rispetto all'indicatrice, si può calcolare la grandezza dei raggi comuni. Misurato l'angolo che essi fanno coi semiassi delle corrispondenti ellissi sezioni principali dell'indicatrice, si possono poi calcolare questi che sono, a due a due, gli indici principali. Alla determinazione degli indici si può anche giungere mediante misure di ritardo od usando la relazione di Mallard.

ABSTRACT. — Each intersection between the plane of a random section of a biaxial crystal and one of the principal planes of its indicatrix is a radius both of the section-plane ellipse and a principal elliptic section of the indicatrix. If the orientation of the section with regard to the indicatrix and the refractive indices of the section are known, the length of each common radius can be calculated. Then, since the angle between radius and semiaxes of the corresponding principal section of the indicatrix can be measured, the length of these semiaxes, which are, two by two, principal refractive indices, can be calculated.

The determination can also be obtained by means of retardation measurements, or by employing Mallard's relation.

Special U-stage and accessories, a cumbersome outfit and complicated manipulations and observations are necessary to determine directly with the polarizing microscope the three principal refractive indices of biaxial crystal grains or thin sections: on the contrary, both indices of a randomly oriented grain or section can be determined much more easily and quickly.

The purpose of the present paper is to show that the three principal indices can be determined if both the indices $n_{\gamma'}$ and $n_{\alpha'}$ of a random section and the orientation of the section with respect to the optic indicatrix are known.

In fact, in general (particular cases are discussed below), the plane of a crystal section intersects all three principal planes of the indicatrix: in consequence, the intersections are radii both of the ellipse central section of the indicatrix corresponding to the plane of the section, and of each ellipse principal section of the indicatrix.

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Let draw the cyclographic projection of the indicatrix principal planes in the plane of the section (Fig. 1): in it $N_{x1}N_{x1}$, $N_{x2}N_{x2}$, $N_{x3}N_{x3}$ are the directions of the intersections and $E_{\gamma'}$, $E_{\gamma'}$, $E_{\alpha'}$, $E_{\alpha'}$ those of the extinctions of the sections.

Consider first the intersections in so far as radii of the ellipse intersected by the plane of the section. Both the directions ($E_{\gamma'}$ and $E_{\alpha'}$) and the length ($n_{\gamma'}$ and $n_{\alpha'}$) of the semiaxes of the ellipse are known; each angle

$$\Psi_1 = \widehat{E_{\gamma'}N_{x1}}\ , \quad \Psi_2 = \widehat{E_{\gamma'}N_{x2}}\ , \quad \Psi_3 = \widehat{E_{\gamma'}N_{x3}}$$

between each radius and the major axis of the ellipse is found in the course of the determination of the orientation of the indicatrix (difference between the A_1 graduations of $E_{\gamma'}$ and N_x) or can be measured in the projection. Therefore the length n_{x1}, n_{x2}, n_{x3} of the radii can be calculated:

$$n_{x1}^2 = \frac{n_{\gamma'}^2 n_{\alpha'}^2}{n_{\gamma'}^2 \sin^2_{\Psi 1} + n_{\alpha'}^2 \cos^2_{\Psi 1}}\ ; \quad n_{x2}^2 = \frac{n_{\gamma'}^2 n_{\alpha'}^2}{n_{\gamma'}^2 \sin^2_{\Psi 2} + n_{\alpha'}^2 \cos^2_{\Psi 2}}\ ; \quad n_{x3}^2 = \frac{n_{\gamma'}^2 n_{\alpha'}^2}{n_{\gamma'}^2 \sin^2_{\Psi 3} + n_{\alpha'}^2 \cos^2_{\Psi 3}}$$

or obtained by means of a nomogram (WINCHELL, 1931, plate 2; EMMONS, 1943 and 1953, plate 10; BURRI, 1950, plate 2).

Consider now that each radius n_x is at the same time a radius of an ellipse whose semiaxes are two principal refractive indices; the angles between radius and major semiaxis

$$\rho_1 = \widehat{N_{x1}\gamma}\ ; \qquad \rho_2 = \widehat{N_{x2}\gamma}\ ; \qquad \rho_3 = \widehat{N_{x3}\beta}$$

can be measured in the projection.

In each equation of these three ellipses the unknowns are then, two by two, the three principal indices and they can be determined by solving the system of the three equations.

Taking

$$A = \frac{1}{n_\alpha}\ ;\ B = \frac{1}{n_\beta}\ ;\ C = \frac{1}{n_\gamma}\ ;\ D_1 = \frac{1}{n_{x1}}\ ;\ D_2 = \frac{1}{n_{x2}}\ ;\ D_3 = \frac{1}{n_{x3}}$$

the three equations are

$$D_1^2 = B^2 \sin^2_{\rho 1} + C^2 \cos^2_{\rho 1}\ ; \quad D_2^2 = A^2 \sin^2_{\rho 2} + C^2 \cos^2_{\rho 2}\ ; \quad D_3^2 = A^2 \sin^2_{\rho 3} + B^2 \cos^2_{\rho 3}$$

from which

$$A^2 = \frac{D_2^2 - C^2\cos^2_{\rho 2}}{\sin^2_{\rho 2}} = \frac{D_3^2 - B^2\cos^2_{\rho 3}}{\sin^2_{\rho 3}} \tag{1}$$

$$B^2 = \frac{D_1^2 - C^2\cos^2_{\rho 1}}{\sin^2_{\rho 1}} = \frac{D_3^2 - A^2\sin^2_{\rho 3}}{\cos^2_{\rho 3}} \tag{2}$$

$$C^2 = \frac{D_2^2 - A^2\sin^2_{\rho 2}}{\cos^2_{\rho 2}} = \frac{D_1^2 - B^2\sin^2_{\rho 1}}{\cos^2_{\rho 1}} \tag{3}$$

and

$$n_\gamma = \left(\frac{\sin^2_{\rho 1}\cos^2_{\rho 2}\sin^2_{\rho 3} + \cos^2_{\rho 1}\sin^2_{\rho 2}\cos^2_{\rho 3}}{D_1^2\sin^2_{\rho 2}\cos^2_{\rho 3} + D_2^2\sin^2_{\rho 1}\sin^2_{\rho 3} - D_3^2\sin^2_{\rho 1}\sin^2_{\rho 2}}\right)^{\frac{1}{2}} =$$

$$= \frac{n_{x1}n_{x2}n_{x3}\ \ (\sin^2_{\rho 1}\cos^2_{\rho 2}\sin^2_{\rho 3} + \cos^2_{\rho 1}\sin^2_{\rho 2}\cos^2_{\rho 3})^{\frac{1}{2}}}{(n^2_{x2}\sin^2_{\rho 2}\cdot n^2_{x3}\cos^2_{\rho 3} + n^2_{x1}\sin^2_{\rho 1}\cdot n^2_{x3}\sin^2_{\rho 3} - n^2_{x1}\sin^2_{\rho 1}\cdot n^2_{x2}\sin^2_{\rho 2})^{\frac{1}{2}}} \tag{I}$$

$$n_\beta = \left(\frac{\sin^2_{\rho 1}\cos^2_{\rho 2}\sin^2_{\rho 3} + \cos^2_{\rho 1}\sin^2_{\rho 2}\cos^2_{\rho 3}}{D_1^2\cos^2_{\rho 2}\sin^2_{\rho 3} + D_3^2\cos^2_{\rho 1}\sin^2_{\rho 2} - D_2^2\cos^2_{\rho 1}\sin^2_{\rho 3}}\right)^{\frac{1}{2}} =$$

$$= \frac{n_{x1}n_{x2}n_{x3}\ \ (\sin^2_{\rho 1}\cos^2_{\rho 2}\ \sin^2_{\rho 3} + \cos^2_{\rho 1}\sin^2_{\rho 2}\cos^2_{\rho 3})^{\frac{1}{2}}}{(n^2_{x2}\cos^2_{\rho 2}\cdot n^2_{x3}\sin^2_{\rho 3} + n^2_{x1}\cos^2_{\rho 1}\cdot n^2_{x2}\sin^2_{\rho 2} - n^2_{x1}\cos^2_{\rho 1}\cdot n^2_{\rho 3}\sin^2_{\rho 3})^{\frac{1}{2}}} \tag{II}$$

$$n_\alpha = \left(\frac{\sin^2_{\rho 1}\cos^2_{\rho 2}\sin^2_{\rho 3} + \cos^2_{\rho 1}\sin^2_{\rho 2}\cos^2_{\rho 3}}{D_2^2\cos^2_{\rho 1}\cos^2_{\rho 3} + D_3^2\sin^2_{\rho 1}\cos^2_{\rho 2} - D_1^2\cos^2_{\rho 2}\cos^2_{\rho 3}}\right)^{\frac{1}{2}} =$$

$$= \frac{n^2_{x1}n_{x2}n_{x3}\ \ (\sin^2_{\rho 1}\cos^2_{\rho 2}\sin^2_{\rho 3} + \cos^2_{\rho 1}\sin^2_{\rho 2}\cos^2_{\rho 3})^{\frac{1}{2}}}{(n^2_{x1}\cos^2_{\rho 1}\cdot n^2_{x3}\cos^2_{\rho 3} + n^2_{x1}\sin^2_{\rho 1}\cdot n^2_{x2}\cos^2_{\rho 2} - n^2_{x2}\cos^2_{\rho 2}\cdot n^2_{x3}\cos^2_{\rho 3})^{\frac{1}{2}}} \tag{III}$$

Although apparently complicated, the calculations can be carried out in a short time by means of a «scientific» pocket calculator: it takes more time to determine $n_{\gamma'}$ and $n_{\alpha'}$, and to draw the projection.

Although somewhat less accurately, the principal indices can be obtained differently.

In fact, since each common radius n_x lies in a principal plane of the indicatrix, n_x is perpendicular to a third principal axis of the indicatrix, i.e. to a principal index $\mathbf{n}_\zeta$. In consequence, on n_{x1}, n_{x2}, n_{x3} are known, if the birefringence of each virtual section of the indicatrix through both n_x and n_ζ can be determined, the value of the principal indices can be obtained.

The magnitude of the birefringence of the virtual sections can be obtained either: 1) by measuring directly the retardation both of the crystal section and of each virtual section; or: 2) by means of Mallard's relation, if the angle of the optic axes is known.

1. The indices of the section being known, the retardation $R = (n_{\gamma'} - n_{\alpha'}) \cdot t$ of the section is measured and recorded. Then, with the U-stage, after recording the A_1 graduation of the positive extinction of the section, the first operation to determine the orientation of the indicatrix is carried out, i.e., the section is rotated around A_1 and A_2 to set a principal plane of the indicatrix to the vertical position. A virtual section is thus perpendicular to the microscope axis: in fact, one of the common radii n_x is in the NS direction (parallel to A_2) and one of the principal indices $\mathbf{n}_\zeta$ in the EW direction (parallel to A_4). The difference between the A_1 graduation both of n_x and of the positive extinction gives Ψ, so that the length of n_x can be obtained, as above shown. Now the retardation of the virtual section $R' = (n_\zeta - n_x) \cdot t \cdot \cos\chi$ is measured and the ratio R/R' calculated. The birefringence of the virtual section is then:

$$n_\zeta - n_x = \frac{R}{R'} (n_{\gamma'} - n_{\alpha'}) \cdot \cos\chi$$

The angle χ is not known: in fact, the section has been inclined by an angle α_2 around the A_2 axis and, in consequence, the geometrical thickness of the section along the microscope axis is now $t/\cos\alpha_2$, but the incident ray inclined by α_2 to the section plane is refracted in the crystal in two rays, according to both the refractive indices of the virtual section and the refractive index of the U-stage segment. The path of both wave fronts in the crystal section is therefore different both from each other and from the geometrical thickness. However, in the majority of cases, i.e. if the birefringence is not very large, the angle α_2 being not greater than, say, 35°-40°, a single angle χ can be considered, equal to α_2 corrected in the usual way by the difference of refractive indices n_x and $n_{segm.}$, so that $\cos\chi = \cos\alpha_2'$. Each principal index n_ζ can therefore be determined.

2. Alternatively, if 2V is known, after Ψ_1, Ψ_2, Ψ_3 and n_{x1}, n_{x2}, n_{x3} have been determined, the poles P_1, P_2, P_3 of the virtual sections are plotted: obviously, they are on each great circle projection of a principal plane, at its intersection with the diameter through the great circle pole. Then the angles Θ_1 and Θ_2 between each optic axis projection and the crystal section pole are measured in the projection and by means of Mallard's relation the maximum birefringence of the crystal (is calculated

$$n_\gamma - n_\alpha = \frac{n_{\gamma'} - n_{\alpha'}}{\sin_{\theta 1} \sin_{\theta 2}}$$

Then the angles Θ' $(= \Theta'_1 = \Theta'_2)$; Θ''_1, Θ''_2; Θ''' $(= \Theta'''_1 = \Theta'''_2)$ between each axis projection and respectively each pole P_1, P_2, P_3 are measured in the projection. Since the maximum birefringence is known, again by means of Mallard's relation the partial birefringence of each virtual section and, in consequence, the principal refractive indices can be calculated:

$$n_{x1} - n_\alpha = (n_\gamma - n_\alpha) \sin^2\theta'$$

$$\pm n_\beta \mp n_{x2} = (n_\gamma - n_\alpha) \sin\theta''_1 \sin\theta''_2$$

$$n_\gamma - n_{x3} = (n_\gamma - n_\alpha) \sin^2\theta'''$$

Particular cases are if the crystal plane does not intersect all three principal planes i.e. if it is perpendicular to one of them. One of the section indices is then a principal index. If 2V is known it is still possible to determine both remaining principal indices.

If n_γ and $n_{\beta'}$ are the indices of the section, from the angles Θ $(= \Theta_1 = \Theta_2)$ between the projection centre and the projection of the optic axes, the maximum birefringence can be calculated:

$$n_\gamma - n_\alpha = (n_\gamma - n_{\beta'})/\sin^2\theta$$

from wich n_α in obtained; n_β is given by

$$n_\beta = n_\gamma - (n_\gamma - n_\alpha)\sin^2 V_\gamma$$

Likewise if the section indices are $n_{\beta'}$ and n_α

$$n_\gamma = n_\alpha + (n_{\beta'} - n_\alpha)/\sin^2\theta$$
$$n_\beta = n_\alpha + (n_\gamma - n_\alpha)\cos^2 V_\gamma$$

If the section indices are n_β and either $n_{\alpha'}$, or $n_{\gamma'}$, $\Theta_1 \pm \Theta_2 = 2V$. Then if one the optic axes is set parallel to the microscope axis, by rotating around n_β, i.e. around A_4, by α_4, $n_\beta \sin\Theta = n_{segm.}\sin\alpha_4$ and $\sin\Theta = n_{segm.}\sin\alpha_4/n_\beta$ from which 2V and the angles between $n_{\beta'}$ and both n_α and n_γ are known, so that it is possible to calculate the indices or to read their value in a nomogram.

In order to check the reliability and accuracy of the methods suggested, the three principal indices of a 92.5 % An low temperature plagioclase have been calculated from the refractive indices and indicatrix orientation data of (010) and (001) generalized cleavage flakes, given by Burri, Parker and Wenk (1967, p. 309 for the refractive indices and pp. 295 and 297, as well as Plates VII and VIII for the indicatrix and optic axes coordinates).

The Burri et al. data are the following:

Cleavage flake:		(010)	(001)
$n_{\gamma'}$		1.5822	1.5812
$n_{\alpha'}$		1.5761	1.5766
$n_{\gamma'} - n_{\alpha'}$		0.0061	0.0046
n_γ	δ	326.2	99.1
	ε	43.9	39.7
n_β	δ	85.5	210.4
	ε	62.5	73.2
n_α	δ	196.3	312.5
	ε	55.7	55.3
A	δ	241.8	119.6
	ε	29.6	87.5
B	δ	175.7	348.9
	ε	89.8	22.0

$n_\gamma = 1.5853$; n_β not given; $n_\alpha = 1.5730$; $2V = 101.8°$; $n_\gamma - n_\alpha = 0.0123$

Based on these data, the stereograms for (010) (Fig. 2) and (001) (Fig. 3) have been traced, the extinction directions $E_{\gamma'}$, $\mathbf{E}_{\gamma'}$ and $E_{\alpha'}$, $E_{\alpha'}$ being plotted by means of Fresnel's construction. The poles P and the angles Ψ, ϱ, Θ, are also marked in the projections. The angles were measured by means of a 20 cm. diameter Wulff net, and the indices n_x were calculated.

The resulting angles and refractive indices are:

Cleavage flake:		(010)	(001)
Ψ_1; Ψ_2; Ψ_3		12°; 58°; 62°	11°; 66°; 45°.5
ρ_1; ρ_2; ρ_3		55°; 53°; 42°	68°; 52°; 27°.5
θ_1; θ_2		30°; 89°	21°.5; 87°.5
θ'; θ''_1; θ''_2; θ'''		58°.5; 14°; 89°; 53°.5	54°.5; 14°; 88°; 46°.5
n_{x1}		1.5834	1.581
n_{x2}		1.5778 (5)	1.5773 (5)
n_{x3}		1.5774	1.5788
$\underline{\underline{n}}_{\gamma}$	(calcul.)	1.585	1.5844
	(Mallard)	1.5852	1.5854
$\underline{\underline{n}}_{\beta}$	(calcul.)	1.5803	1.5804 (6)
	(Mallard)	1.5807	1.5804
$\underline{\underline{n}}_{\alpha}$	(calcul.)	1.5732	1.5732
	(Mallard)	1.5746	1.5727

(Principal refractive indices calculated by means of equations (1); (2); (3) = (*calcul.*) and obtained by means of Mallard's relation = (*Mallard*)).

The result is in good agreement with Burri et al. data: obviously, the accuracy

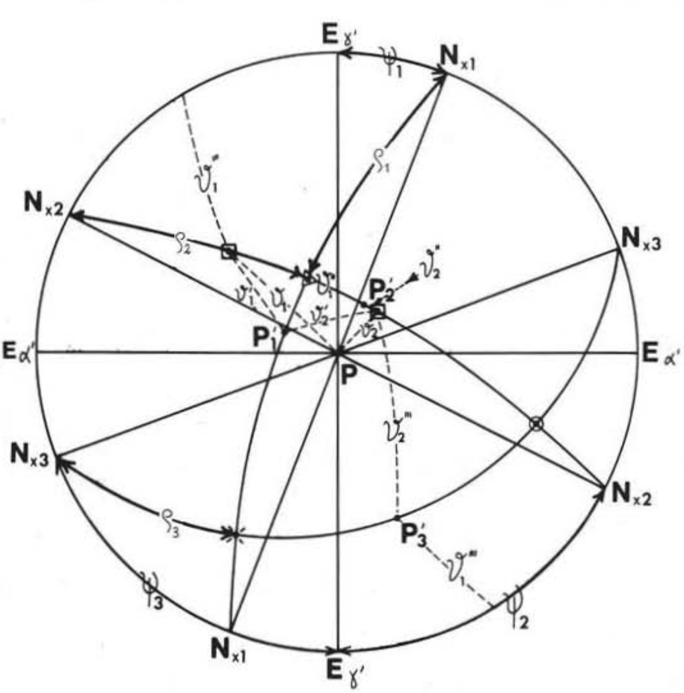
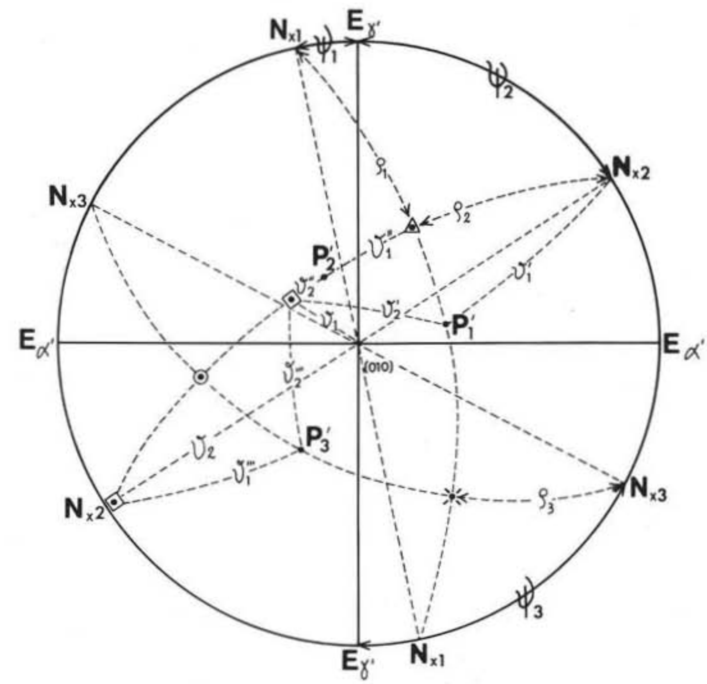
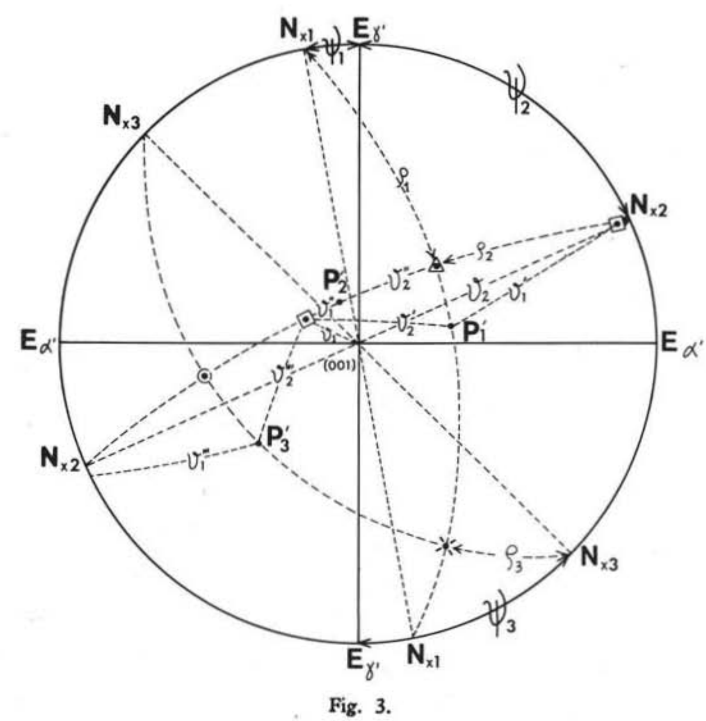


Fig. 1.



N_{x1}
$E_{\gamma'}$
ψ_1
ψ_2
N_{x2}
N_{x3}
ρ_1
ρ_2
P_2'
ϑ_1''
ϑ_2''
ϑ_1'
ϑ_2'
ϑ_1
P_1'
$E_{\alpha'}$
$E_{\alpha'}$
(010)
ϑ_2'''
P_3'
ϑ_2
ϑ_1'''
N_{x2}
ρ_3
N_{x3}
ψ_3
$E_{\gamma'}$
N_{x1}

Fig. 2.



N_{x1}
$E_{\gamma'}$
ψ_1
ψ_2
N_{x3}
ρ_1
N_{x2}
ρ_2
P_2'
ϑ_2''
ϑ_2
ϑ_1'
ϑ_1''
ϑ_2'
P_1'
ϑ_1
$E_{\alpha'}$
$E_{\alpha'}$
(001)
ϑ_2'''
P_3'
N_{x2}
ϑ_1'''
ρ_3
N_{x3}
ψ_3
$E_{\gamma'}$
N_{x1}

Fig. 3.

in the determination depends upon the accuracy in measuring both the indices of the section and the angles Ψ and ϱ, and upon the magnitude itself of these angles.

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ERRATA CORRIGE

Vol. XXXIII (1977) Fascicolo 2

A. DEBENEDETTI - Indirect determination etc.

p. 541, formula (III)

at the numerator, instead of n^2_{x1}, read n_{x1};

at the denominator, instead of n_{x3}, read n^2_{x3};

p. 542, line 8: instead of on, read once.

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