On changing the plane of a gnomonic or stereographic projection.

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§ 1. CONSIDER the poles of the faces of a crystal and the zones (great circles) which join them. Suppose we have given their gnomonic projection on to a certain tangent-plane to the fundamental sphere on which these poles and zones lie. We can find the gnomonic projection on any other plane by aid of the gnomonic net, as explained in this Magazine (1904, vol. xiv, p. 19); and similarly for the stereographic or orthographic projection.

A means of effecting the change of plane when a net is not available was given in this Magazine (1917, vol. xviii, p. 127) for the case of the orthographic projection. Here we give a method which may be employed in the case of the gnomonic projection.

For the orthographic projection we found the new positions of the projections of face-poles and edge-poles when the sphere is rotated about a diameter, the plane of projection being fixed (which amounts to the same as keeping the sphere fixed and altering the plane of projection). This is troublesome for the gnomonic projection, but we may effect our purpose by finding the new positions of the projections of the *zones*; since the projections of face-poles and edge-poles are at once deduced.[1]

In addition to a ruler and set-square we require only a *tangent-scale*. This is a straight-edge marked with degrees, so that the division corresponding to an angle θ is at a distance $r \cdot \tan\theta$ from the zero of the scale; r being the radius of the sphere. Such a tangent-scale is made very quickly out of a strip of squared paper with the help of a table of natural tangents. It may conveniently be divided so as to show every 2°.

[1] For a face-pole is the intersection of two zones, &c. V. Goldschmidt gave a method of finding the *directions* of the new projections in Zeits. Kryst. Min., 1891, vol. xix, p. 352. See also J. W. Evans, this Magazine, 1906, vol. xiv, p. 151.

Suppose then we have the projection of the zones (O being the centre of the projection), and we require the new projection when the sphere has been rotated about the diameter parallel to the line λ through an angle ϕ towards the reader (in fig. 1 $\phi = 70°$). Place the tangent-scale as shown in the figure with its zero at O and its edge μ perpendicular to λ. Draw the lines e, f parallel to λ and at a distance $\frac{1}{2}\phi$ from λ as shown by the tangent-scale. Let the original projection of any zone meet e in B and μ in A. Take D on f so that BD is perpendicular to λ, and C on μ so that AC is ϕ (as measured by the tangent-scale). Then CD is the new projection of the zone. The reason is at once clear on considering the gnomonic net. Similarly we can deal with each zone.

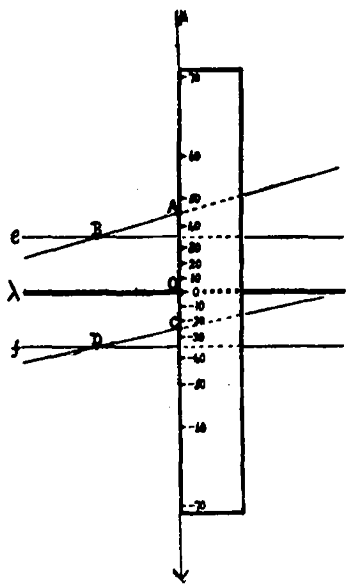

Fig. 1.

A difficulty may arise if A or B is at an inconvenient distance from O. But in practice it will be found that the difficulty is readily surmounted. For instance, suppose A and C on the paper, but B or D too far away. We may find that the original projections of two other zones meet on AB. Then if the new projections of these two zones meet at P, PC is the new projection of AB.

In the last resort the new projection of any point, whose original projection is R, can be found as follows. Let RH parallel to λ meet μ in H. Take along μ $HJ = \phi$ and $HV = \frac{1}{2}(\phi + 180°)$, as given by the tangent-scale. Then VR meets the line J parallel to λ at the new projection S of R. But this process will seldom be required.

If we take R on AB so that the new projection of RH is at infinity (i.e. $OH = 90° - \phi$, $VO = \frac{1}{2}\phi$), then VR is parallel to CD. This is Goldschmidt's method referred to above.

§ 2. When we have the gnomonic projection of the zones on any plane, the orthographic picture of the crystal on the plane may be drawn.[1] For each edge in the picture of the crystal is perpendicular to the gnomonic projection of the corresponding zone. If we have the orthographic picture on the original plane, we get the orthographic picture on the new plane by forming the new gnomonic projections of the zones (which are perpendicular to the new orthographic pictures of the edges[2]), and noting that the line joining corresponding points in the two orthographic pictures is perpendicular to λ. (See fig. 4, this Magazine, 1917, vol. xviii, p. 128.)

§ 3. The following process is available for changing the plane when the projection is stereographic, instead of gnomonic.

Suppose, as before, the sphere rotated about the diameter λ through an angle ϕ; the plane of projection being kept fixed. Let f be the fundamental circle of the projection, with centre O and radius r. We replace our tangent-scale by a 'tangent-half-scale', i.e. a straight-edge in which the division corresponding to an angle θ is at a distance $r \, . \tan \frac{1}{2}\theta$ from the zero of the scale (not $r \, . \tan \theta$, as in § 1). It is placed with its zero at O and its edge along μ perpendicular to λ.

Let e be the zone of the old projection which becomes f in the new; so that e and f have λ as radical axis and cut at an angle ϕ ($\phi = 35°$ in fig. 2). The centre of e is on μ at a distance $r \, . \tan \phi$ from O (2ϕ as shown by the scale). Let V be the centre of the circle coaxial with e and f, and bisecting the angle between them. The distance VO is $r \, . \tan \frac{1}{2}\phi$ (ϕ as shown by the scale). The point V is the projection of the pole of the zone whose projection is e.

Suppose now that the original projection of any zone meets μ at A and e at B. Make $AC = \phi$, as shown by the scale, and let VB cut f at D. Then the new projection of the zone passes through C, D and has

[1] That is, the picture of the crystal as seen by an eye at a considerable distance in the normal to the plane.

[2] The *directions* of the new projections of the zones are sufficient for drawing the picture of the crystal. See Goldschmidt, loc. cit.

its centre on the line through O perpendicular to OD. Hence it is readily drawn.

§ 4. If we require merely the orthographic picture of the crystal on the new plane of projection, it is unnecessary to draw completely the new projections of the zones. In fact, the picture of the edge corresponding to the zone CD is perpendicular to OD. Hence all we require

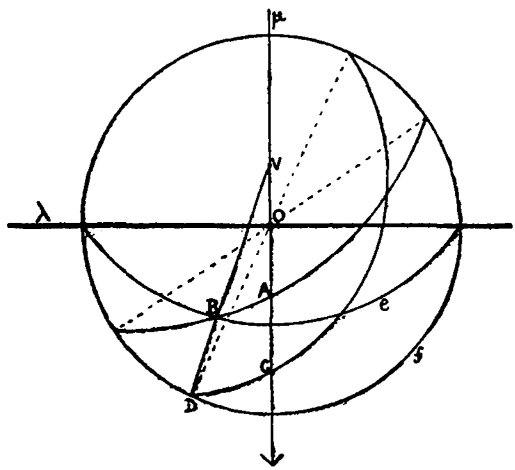

FIG. 2.

is the position of D; and we need not concern ourselves with the points A and C.

This process has a considerable analogy with Goldschmidt's method for the gnomonic projection, as the reader will readily see, on comparing his paper with the author's 'Mathematical Crystallography', 1903, chap. i, § 5.