On cleavage-angle in a random section of a crystal.

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§ 1. SUPPOSE a crystal has two good cleavages, and that a thin section is cut in a random direction. Cleavage-cracks are seen in the section which consist of two sets of parallel lines inclined at an acute angle ϕ, which may have any value whatever, depending on the direction in which the section is cut. The question suggests itself: what is the chance that ϕ may lie between assigned limits? This question proves to be incapable of a direct mathematical solution, but it can be solved graphically. This is worked out here for two typical cases, and the general method of performing the graphical solution in every case is indicated.

§ 2. Suppose that a sphere is taken, and that the radii of the sphere perpendicular to the cleavage-planes meet the sphere in A and B. Let O be the pole of the great circle AB. Suppose that the radius perpendicular to the plane of the random section meets the sphere in P. It is readily seen that the acute angle ϕ between the cleavage-cracks is the acute angle between the great circles PA and PB. If $\phi = a$, P traces out a curve on the sphere. The ratio of the area enclosed by this curve to the area of the sphere is the chance that the angle in a random section should be $\leq a$.

To find the curves, suppose $AB = 2\beta$, $OP = \rho$, and that OP makes an angle θ with the bisector of the angle AOB. Then from the triangles OPA, OPB, APB

$$\cos AP = \sin\rho \,.\, \cos(\beta-\theta), \qquad \cos BP = \sin\rho \,.\, \cos(\beta+\theta),$$
$$\cos 2\beta = \cos AP \,.\, \cos BP \pm \sin AP \,.\, \sin BP \,.\, \cos a.$$

Eliminating AP, BP, we have

$$\{\cos 2\beta - \cos 2\theta + 2\cos 2\beta \,.\, \cot^2\rho\}^2 = 4\sin^2 2\beta \,.\, \cot^2 a\,(1+\cot^2\rho)$$

and

$$2\cos 2\beta \,.\, \operatorname{cosec}^2\rho \pm 2\sin 2\beta \,.\, \cot a \,.\, \operatorname{cosec}\rho - (\cos 2\beta + \cos 2\theta) = 0.$$

Assigning different values to θ we get the corresponding values of $\cot\rho$ or $\operatorname{cosec}\rho$, and thus trace the curves.

In figs. 1 and 2 we map the curves for the cases $2\beta = 90°$ and $2\beta = 60°$, which we shall call respectively the 'augite' and 'hornblende' cases. The plane map is drawn so that the polar co-ordinates of P are $(\sin\frac{1}{2}\rho,\ \theta)$. This mapping causes considerable distortion in the curves, but secures the equality of corresponding areas on the sphere and the map.

In fig. 1 we have $2\beta = 90°$. The curves corresponding to the following values of a are shown in order as we go out from O horizontally or vertically: $a = 90°$ (the lines OA and OB), $a = 76°$ ($\tan^{-1}4 = 75°\,58'$), $a = 60°$, $a = 45°$, $a = 30°$, $a = 0°$ (a circle with centre O).

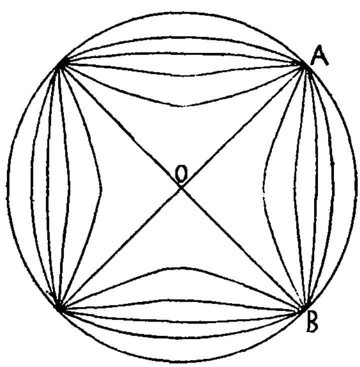

FIG. 1.—$2\beta = 90°$.

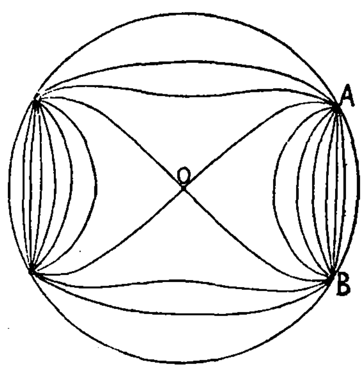

FIG. 2.—$2\beta = 60°$.

In fig. 2 we have $2\beta = 60°$. As we go out from O horizontally we have $a = 60°$ (passing through O), $a = 76°$, $a = 90°$ (touching OA, OB at A and B), $a = 76°$, $a = 60°$, $a = 45°$, $a = 30°$, $a = 0°$ (a circle). As we go out from O vertically we have $a = 60°$, $a = 45°$, $a = 30°$, $a = 0°$.

§ 3. Measuring the area enclosed by any one of these curves (drawn on squared millimetre paper, and the squares counted), we get fig. 3, in which is shown the chance p that the cleavage-crack angle ϕ for a given random section should be $\leq a$. The broken line refers to the hornblende case and the continuous line to the augite. The horizontal scale shows the value of a (every 10°) and the vertical scale the value of p (every 0·1).

From the figure may be read off any information we require. For instance, the chance that ϕ lies between 50° and 70° is about 0·36 and 0·21 in the hornblende and augite cases respectively.

§ 4. In fig. 4 we show the slope (the tangent of the angle which the tangent makes with the horizontal) of the two curves of fig. 3. The broken and continuous lines refer to the hornblende and augite cases respectively; the horizontal scale shows a (every 10°) and the vertical scale the slope (every unit).

Suppose that in a rock-section a mineral is known to be either hornblende or augite, the a priori probabilities of the two cases being estimated as in the ratio $\lambda : \mu$ before the cleavage-crack angle ϕ is measured. Suppose now that ϕ is measured and found to be equal to a. Measure the ordinates in fig. 4 corresponding to the abscissa a. If these are y_1 and y_2, the chance that the mineral is hornblende is

$$\lambda y_1 \div (\lambda y_1 + \mu y_2).$$

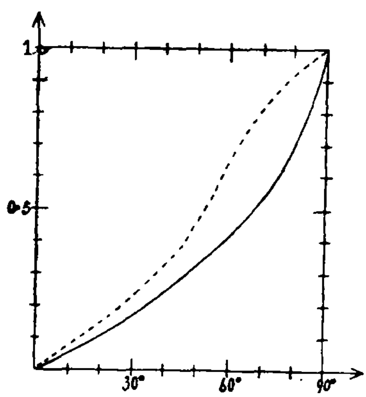

Fig. 3.

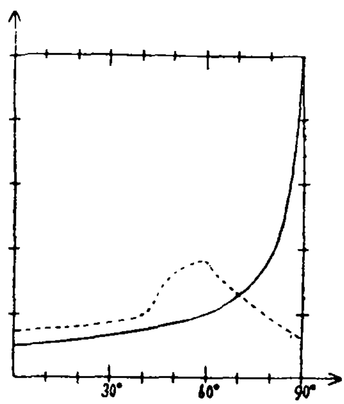

Fig. 4.

For instance, suppose that a priori the mineral is equally likely to be hornblende or augite. If ϕ is nearly 90°, there are long odds (about 8 to 1) in favour of the mineral being augite. If ϕ is 60°, there are odds of about 7 to 4 in favour of its being hornblende. If ϕ is 70°, it is about an even chance that the mineral is hornblende or augite. (In reality ϕ is a little less than this, as the values of 2β in the case of augite and hornblende are 87° 12′ and 55° 49′, not 90° and 60° as has been taken for convenience of calculation. But the above is probably near enough for all practical purposes.)

§ 5. A similar problem is the following: a section of a crystal is taken of given small thickness in a random direction. The birefringence for parallel light normal to the section (as shown by the interferencetint) is a fraction k of the maximum possible birefringence; i.e. that in which the plane of the plate contains the optic axes. What is the chance that $k \leq m$, where m is given?

If the optic axes are the radii of the sphere through A and B, the plane is perpendicular to the radius through P, and the birefringence is a fraction m of the maximum,

$$\cos AP = \sin \rho \,.\, \cos(\beta - \theta), \qquad \cos BP = \sin \rho \,.\, \cos(\beta + \theta),$$
$$\sin AP \,.\, \sin BP = m;$$

provided that the maximum birefringence is not too large.

Then eliminating AP and BP as before, P lies on a curve whose equation is

$$\cos^2 \rho \,(\cos 2\beta + \cos 2\theta)^2$$
$$= -(\sin^2 2\beta + \sin^2 2\theta) + 2\{\sin^2 2\beta \,.\, \sin^2 2\theta + m^2 (\cos 2\beta + \cos 2\theta)^2\}^{\frac{1}{2}}.$$

The curves for various values of m and 2β may be mapped as before. The chance that $k \leq m$ is the ratio of the area enclosed by the corresponding curve to the area of the sphere.

We leave the graphical solution to the reader. In two cases the ratio of the areas is obtainable without recourse to graphical methods.

In a uniaxial crystal the birefringence is $\sin^2 \omega$ times the maximum, ω being the angle between the optic axis and the normal to the section. Hence the chance that the birefringence is $\geq m$ times the maximum is $(1-m)^{\frac{1}{2}}$.

If the optic axes are perpendicular ($2\beta = 90°$), and $m = \frac{1}{2}$, the ratio of the areas is

$$2 \tan^{-1} \frac{10\sqrt{2}}{23} \div \pi, \text{ i.e. about } 0{\cdot}35.$$

Thus the chance that the birefringence is $\geq$ half the maximum when a random section of given thickness is taken in a crystal (i) which is uniaxial, (ii) which has perpendicular optic axes, is respectively 0·71 and 0·65. For any other crystal the chance will lie between these two quantities.