Angular relations between equivalent planes and distances between equivalent points in symmetrical point groups

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A FUNDAMENTAL problem arising out of the study of symmetrical point groups can be formulated as follows: Let N points in three-dimensional space be connected two and two in every possible way by straight lines: what relations must exist between the lengths of these lines if the points are equivalent members of a symmetrical point group?

The methods of projection used in crystallography (e.g. the stereographic projection) at once show that the points may be considered to be the images of straight lines or planes. The problem formulated above has therefore a direct application to descriptive crystallography and can in this connexion be stated as follows:

A complex of N planes is determined by the points of intersection of the normals with the surface of a unit sphere, i.e. by their *poles*. What now must the angular relations between the planes be if all are equivalent, that is to say, belong to one and the same simple form?

The connecting lines between the points are in this case replaced by the angles between the normals of the planes, and in treating this problem it will be useful to substitute these angles by their characteristic cosine values.

If we proceed from any one plane and determine the angles between its normal and those of the $(N-1)$ other planes, it is obvious that these will only be equivalent to the first if certain quite specific relations exist between the angles (or their cosines). These relations we desire to express in formulae.

If all N planes are equivalent, it is, of course, immaterial which we select as the point of departure. Let the planes be numbered from 1 to N and the cosines of the angles between all possible pairs of planes be written in the form of a square matrix. Every row and every column of the matrix must now contain all the cosine values which differ from one another. When written in the usual way, the matrix possesses symmetrical structure in respect to the chief diagonal which contains the values $\cos 0° = 1$. With this fundamental condition others are associated which determine the symmetry and the special character of the form constituted by the equivalent planes.

The position of a pole in respect to the elements of symmetry passing through the centre of the sphere determines the position of the remaining poles and therefore also the angles between the planes. Let the position of the first pole be expressed in terms of the usual co-ordinate angles ϕ and ρ. This is shown in fig. 1 for the complex of planes constituting a dihexagonal pyramid. Our problem now consists in expressing the cosines of the angles between plane 1 and the $(N-1)$ other planes in terms of the ϕ- and ρ-values of the plane of departure.

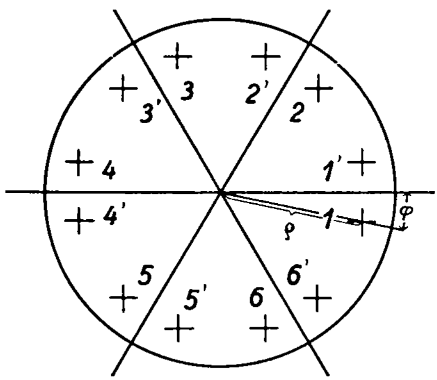

FIG. 1.

In order to obtain the matrix in an easily defined form, it is proposed to use the groupings into cycles commonly employed in the investigation of symmetry. The study of point symmetry leads, as is well known, to the distinction of three principal cases: (1) symmetries with a unique axis (including the orthorhombic, monoclinic, and triclinic symmetries as trivial specializations); (2) the isometric cubic symmetries; (3) the isometric symmetries.

1. *Matrix representation of symmetries with a unique axis.*

In symmetries of this sort we select the highest rotation cycles and proceed to number the poles anti-clockwise from 1 to n, n being the valency of the axis. The unique axis as the rotation axis with the highest valency having thus the valency n, the number of equivalent elements in the derived symmetry groups can at most be $4n = N$. For instance, n planes of symmetry parallel to the unique axis, or n binary axes perpendicular to the same, or a centre of symmetry, or a plane of symmetry perpendicular to the unique axis may be present. If $N = 4n$, which implies that the group has holohedral character, the total matrix may be resolved into 4×4 submatrices comprising $n \times n$ constituents.

Let the n poles required to augment an n-gonal pyramid to a di-n-gonal one be termed $1' \ldots x' \ldots n'$, the count being taken in the same anti-clockwise sense as hitherto (see fig. 1). Further let the mirror images of $1 \ldots x \ldots n$ with respect to planes of symmetry perpendicular to the unique axis be called $(1) \ldots (x) \ldots (n)$ and those of $1' \ldots x' \ldots n'$ bear the numbers $(1') \ldots (x') \ldots (n')$. For the dihexagonal bipyramid, for

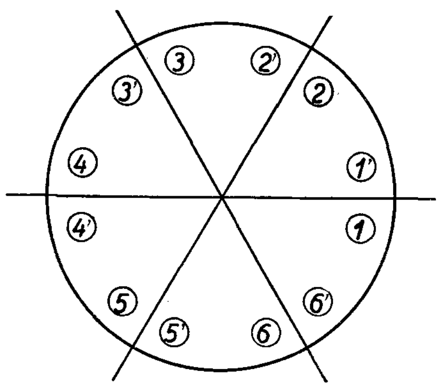

FIG. 2.

instance, fig. 2 with its poles on the lower half of the sphere must now be considered in conjunction with fig. 1. A summary of the possible forms resulting from $n = 6$ can at once be given as follows:

1 2 3 4 5 6 = the hexagonal pyramid or when $\rho = 90°$ the hexagonal prism.

1 2 3 4 5 6 together with 1′ 2′ 3′ 4′ 5′ 6′ = the dihexagonal pyramid or when $\rho = 90°$ the dihexagonal prism.

1 2 3 4 5 6 together with (1) (2) (3) (4) (5) (6) = the hexagonal bipyramid.

1 2 3 4 5 6 together with (1′) (2′) (3′) (4′) (5′) (6′) = the hexagonal trapezohedron.

1 2 3 4 5 6 together with 1′ 2′ 3′ 4′ 5′ 6′ and (1) (2) (3) (4) (5) (6) and (1′) (2′) (3′) (4′) (5′) (6′) = the dihexagonal bipyramid.

If the n of a rotation cycle be even, the cycle can be used to derive the $n/2$-gonal classes of symmetry. We obtain for instance:

1 3 5 = the trigonal pyramid or when $\rho = 90°$ the trigonal prism.

1 3 5 (1) (3) (5) = the trigonal bipyramid.

1 3 5 1′ 3′ 5′ = the ditrigonal pyramid or when $\rho = 90°$ the ditrigonal prism.

1 3 5 1′ 3′ 5′ (1) (3) (5) (1′) (3′) (5′) = the ditrigonal bipyramid.

1 3 5 (2) (4) (6) the rhombohedron.

1 3 5 1′ 3′ 5′ (2) (4) (6) (2′) (4′) (6′) the ditrigonal scalenohedron, etc.

With $\rho = 0$ the formulae will become simpler and lead to forms consisting of one plane (pedion) or two planes (pinacoids) respectively.

In a matrix comprising the cosines appropriate to any given n, all forms pertaining to a symmetry class with a unique axis are, therefore, characterized by the angles between the various planes of the form. This is true whether the class of symmetry is a crystallographically possible one or not. A general representation of such a matrix of cosine values is given in table I a.

TABLE I a. Symbols of the cosine values occurring in the holohedral classes of symmetry with a unique axis.

	1	2	…	$n-1$	n	1′	2′	…	$(n'-1)$	n'	(1)	(2)	…	$(n-1)$	(n)	(1′)	(2′)	…	$(n'-1)$
1	1	α_1	…	$\alpha_{\bar{2}}$	$\alpha_{\bar{1}}$	β_0	β_1	…	$\beta_{\bar{2}}$	$\beta_{\bar{1}}$	a_0	a_1	…	$a_{\bar{2}}$	$a_{\bar{1}}$	b_0	b_1	…	$b_{\bar{2}}$
	(α_0)																		
2	$\alpha_{\bar{1}}$	1	…	$\alpha_{\bar{3}}$	$\alpha_{\bar{2}}$	$\beta_{\bar{1}}$	β_0	…	$\beta_{\bar{3}}$	$\beta_{\bar{2}}$	$a_{\bar{1}}$	a_0	…	$a_{\bar{3}}$	$a_{\bar{2}}$	$b_{\bar{1}}$	b_0	…	$b_{\bar{3}}$
⋮			I					II					III					IV	
$n-1$	α_2	α_3	…	1	α_1	β_2	β_3	…	β_0	β_1	a_2	a_3	…	a_0	a_1	b_2	b_3	…	b_0
n	α_1	α_2	…	$\alpha_{\bar{1}}$	1	β_1	β_2	…	$\beta_{\bar{1}}$	β_0	a_1	a_2	…	$a_{\bar{1}}$	a_0	b_1	b_2	…	$b_{\bar{1}}$
1′	β_0	$\beta_{\bar{1}}$	…	β_2	β_1	1	α_1	…	$\alpha_{\bar{2}}$	$\alpha_{\bar{1}}$	b_0	$b_{\bar{1}}$	…	b_2	b_1	a_0	a_1	…	$a_{\bar{2}}$
2′	β_1	β_0	…	β_3	β_2	$\alpha_{\bar{1}}$	1	…	$\alpha_{\bar{3}}$	$\alpha_{\bar{2}}$	b_1	b_0	…	b_3	b_2	$a_{\bar{1}}$	a_0	…	$a_{\bar{3}}$
⋮			II′					I					IV′					III	
$n'-1$	$\beta_{\bar{2}}$	$\beta_{\bar{3}}$	…	β_0	$\beta_{\bar{1}}$	α_2	α_3	…	1	α_1	$b_{\bar{2}}$	$b_{\bar{3}}$	…	b_0	$b_{\bar{1}}$	a_2	a_3	…	a_0
n'	$\beta_{\bar{1}}$	$\beta_{\bar{2}}$	…	β_1	β_0	α_1	α_2	…	$\alpha_{\bar{1}}$	1	$b_{\bar{1}}$	$b_{\bar{2}}$	…	b_1	b_0	a_1	a_2	…	$a_{\bar{1}}$
(1)	a_0	a_1	…	$a_{\bar{2}}$	$a_{\bar{1}}$	b_0	b_1	…	$b_{\bar{2}}$	$b_{\bar{1}}$	1	α_1	…	$\alpha_{\bar{2}}$	$\alpha_{\bar{1}}$	β_0	β_1	…	$\beta_{\bar{2}}$
(2)	$a_{\bar{1}}$	a_0	…	$a_{\bar{3}}$	$a_{\bar{2}}$	$b_{\bar{1}}$	b_0	…	$b_{\bar{3}}$	$b_{\bar{2}}$	$\alpha_{\bar{1}}$	1	…	$\alpha_{\bar{3}}$	$\alpha_{\bar{2}}$	$\beta_{\bar{1}}$	β_0	…	$\beta_{\bar{3}}$
⋮			III					IV					I					II	
$(n-1)$	a_2	a_3	…	a_0	a_1	b_2	b_3	…	b_0	b_1	α_2	α_3	…	1	α_1	β_2	β_3	…	β_0
(n)	a_1	a_2	…	$a_{\bar{1}}$	a_0	b_1	b_2	…	$b_{\bar{1}}$	b_0	α_1	α_2	…	$\alpha_{\bar{1}}$	1	β_1	β_2	…	$\beta_{\bar{1}}$
(1′)	b_0	$b_{\bar{1}}$	…	b_2	b_1	a_0	a_1	…	$a_{\bar{2}}$	$a_{\bar{1}}$	β_0	$\beta_{\bar{1}}$	…	β_2	β_1	1	α_1	…	$\alpha_{\bar{2}}$
(2′)	b_1	b_0	…	b_3	b_2	$a_{\bar{1}}$	a_0	…	$a_{\bar{3}}$	$a_{\bar{2}}$	β_1	β_0	…	β_3	β_2	$\alpha_{\bar{1}}$	1	…	$\alpha_{\bar{3}}$
⋮			IV′					III					II′					I	
$(n'-1)$	$b_{\bar{2}}$	$b_{\bar{3}}$	…	b_0	$b_{\bar{1}}$	a_2	a_3	…	a_0	a_1	$\beta_{\bar{2}}$	$\beta_{\bar{3}}$	…	β_0	$\beta_{\bar{1}}$	α_2	α_3	…	1
(n')	$b_{\bar{1}}$	$b_{\bar{2}}$	…	b_1	b_0	a_1	a_2	…	$a_{\bar{1}}$	a_0	$\beta_{\bar{1}}$	$\beta_{\bar{2}}$	…	β_1	β_0	α_1	α_2	…	$\alpha_{\bar{1}}$

Let the cosines of the angles between the poles 1 and x in the series $1. . .x. . .n$ be symbolized by α_{x-1}. If the angle between the poles 1 and x is greater when measured anti-clockwise, the expression α_{x-1-n} is used in order always to operate with the cosine of the smaller angle. Thus $\alpha_{n-1} = \alpha_{\bar{1}}$, $\alpha_{n-2} = \alpha_{\bar{2}}$, etc.

Similarly, the cosines of the angles between the poles 1 and x' are symbolized by $\beta_{x'-1}$ or $\beta_{x'-1-n}$, those of the angles between 1 and (x) by a_{x-1} or a_{x-1-n}, and, finally, those of the angles between 1 and (x') by b_{x-1} or b_{x-1-n}.

If n is even, the angles between 1 and $n/2$ or 1 and $n'/2$ or 1 and $(n/2)$ or 1 and $(n'/2)$ separate the positive and negative index values. Assuming all points or planes to be equivalent, every row or column of a submatrix

$$||\alpha_{ik}|| \quad \text{or} \quad ||a_{ik}|| \quad \text{or} \quad ||\beta_{ik}|| \quad \text{or} \quad ||b_{ik}||$$

evidently contains the same number of and at most n different cosine values. Also the relation obtains

$$\alpha_i = \alpha_{\bar{i}} \text{ (e.g. } \alpha_2 = \alpha_{\bar{2}}) \quad \text{and} \quad a_i = a_{\bar{i}} \text{ (e.g. } a_1 = a_{\bar{1}}).$$

The square submatrices I and III which each appear four times in the holohedral general matrix are thus symmetrical in themselves. Therefore when n is even, the subsquares I can by their very nature contain at most $n/2$ different cosine values. These include α_0 representing the value of $\cos 0^\circ = 1 = \cos^2\rho + \sin^2\rho$ and $\alpha_{n/2}$ representing the value of $\cos 2\rho = \cos^2\rho - \sin^2\rho$. If n is odd, the last-named value does not occur and the number of different cosine values is $1+(n-1)/2$.

In the subsquare III the number of different cosine values is the same. However, $a_0 = \cos(\sphericalangle\, 1 \text{ to } (1)) = \cos^2\rho + \sin^2\rho$ and $a_{n/2} = -1$. The latter value does not occur when n is odd. Quite generally

$$\alpha_i = \alpha_{\bar{i}} = \cos^2\rho + \sin^2\rho . \cos\frac{i.360}{n}$$

and

$$a_i = a_{\bar{i}} = -\cos^2\rho + \sin^2\rho . \cos\frac{i.360}{n}.$$

In the submatrices I the sum of all cosine values belonging to one row or column (i.e. $\sum \alpha_i$) is given by $n \cos^2\rho$, for the sum $\cos\{(i.360/n)\}$ of the angles derived from one rotation axis is always zero. Similarly, the sum $\sum a_i$ of any row or column is $-n \cos^2\rho$.

The submatrices or subsquares containing the n_β or n_b values each give rise to two matrices, namely, II and II′ and IV and IV′ respectively, of which the primed ones are conjugated to the non-primed. The submatrices II, II′, and IV, IV′ each contain at most n different cosine values which occur once in each row and column.

$$\beta_0 \text{ is } \cos^2\rho + \sin^2\rho \cos 2\phi$$

and when n is even

$$\beta_{n/2} = \beta_{\bar{n}/2} = \cos^2\rho - \sin^2\rho \cos 2\phi,$$

b_0 is $-\cos^2\rho+\sin^2\rho\cos 2\phi$ and when n is even,

$$b_{n/2} = b_{\overline{n}/2} = -\cos^2\rho-\sin^2\rho\cos 2\phi.$$

β_i no longer equals $\beta_{\bar{i}}$, nor is $b_i = b_{\bar{i}}$. The values of these expressions are now given by

$$\beta_i = \cos^2\rho+\sin^2\rho\cos\left(\frac{i.360}{n}+2\phi\right);$$

$$\beta_{\bar{i}} = \cos^2\rho+\sin^2\rho\cos\left(\frac{i.360}{n}-2\phi\right);$$

$$b_i = -\cos^2\rho+\sin^2\rho\cos\left(\frac{i.360}{n}+2\phi\right);$$

$$b_{\bar{i}} = -\cos^2\rho+\sin^2\rho\cos\left(\frac{i.360}{n}-2\phi\right).$$

It can, however, easily be shown that the following relations still obtain:

Sum of all the cosine values in any row or column of II or II′

$$= n\cos^2\rho.$$

Sum of all the cosine values in any row or column of IV or IV′

$$= -n\cos^2\rho$$

These conditions and the arrangement of the minor squares show that the general holohedral matrix possesses centrosymmetrical structure. The chief diagonals contain only ones and the trace has the value N. Table I *b* gives a summary of these results.

TABLE I *b*.

General formulae for the cosine values.	Specal case for identity.	Special case applicable only when n is even.
$\alpha_{\pm i} = \cos^2\rho+\sin^2\rho\cos i\frac{2\pi}{n}$	$\alpha_0 = 1$	$\alpha_{n/2} = \cos^2\rho-\sin^2\rho$
$\beta_{\pm i} = \cos^2\rho+\sin^2\rho\cos\left(i\frac{2\pi}{n}\pm 2\phi\right)$	$\beta_0 = \cos^2\rho+\sin^2\rho\cos 2\phi$	$\beta_{n/2} = \cos^2\rho-\sin^2\rho\cos 2\phi$
$a_{\pm i} = -\cos^2\rho+\sin^2\rho\cos i\frac{2\pi}{n}$	$a_0 = -\cos^2\rho+\sin^2\rho$	$a_{n/2} = -1$
$b_{\pm i} = -\cos^2\rho+\sin^2\rho\cos\left(i\frac{2\pi}{n}\pm 2\phi\right)$	$b_0 = -\cos^2\rho+\sin^2\rho\cos 2\phi$	$b_{n/2} = -\cos^2\rho-\sin^2\rho\cos 2\phi$

If the arrangement of the points or planes n is centrosymmetrical, each positive cosine value requires the presence of an equal negative cosine value. When n is even, the corresponding values may be expressed as

$$\cos^2\rho+\sin^2\rho\cos\frac{i.360}{n} \quad \text{and} \quad -\cos^2\rho+\sin^2\rho\cos\frac{i'.360}{n}$$

in which $i' = n/2+i$. We now, however, write $\bar{i}$ instead of i' and $\bar{i} = n/2-\bar{i}$. Thus with $n = 6$ we obtain the value

$$\cos\frac{1.360}{6} = -\cos\frac{(3+1).360}{6} = -\cos\frac{2.360}{6}.$$

Therefore $\alpha_{\bar{2}} = -\alpha_1$.

Let the total number of different cosine values be Z and the number of cosines different in absolute value only (i.e. irrespective of signs) be z. The following relations can now be established in the series.

n divisible by 4. Beside 1, $\bar{1}$, 0, 0 the other values appear twice with positive and twice with negative signs. z is therefore given by

$$\frac{n-4}{4}+2 = \frac{n+4}{4}.$$

For instance, in the case of a 12-fold axis the $\cos\{(i.360)/12\}$ are as follows:

1 0·866 0·5 0 −0·5 −0·866 −1 −0·866
−0·5 0 0·5 0·866.

n only divisible by 2. Beside 1 and $\bar{1}$ every other cosine value appears twice with positive and twice with negative sign. The total number of numerical values $\cos\{(i.360)/n\}$ (irrespective of signs) is

$$z = \frac{n-2}{4}+1 = \frac{n+2}{4}.$$

For the series $\cos\{(i.360)/6\}$ the values are:

1 0·5 −0·5 −1 −0·5 0·5.

n odd. Beside 1 every other cosine value appears twice with the same sign. Although a given positive numerical value no longer leads to the corresponding negative numerical value, the expression $\sum\cos\{(i.360)/n\}$ (in which i varies from 0 to $n-1$) remains zero.

As a result of the conditions contained in the above formulae, the entire range of cosine values contained in the matrix consists of functions of ρ, ϕ, or of these and 2ϕ. Naturally the following relations obtain:

$$\cos\left(\frac{i.360}{n}+2\phi\right) = \cos\frac{i.360}{n}\cos 2\phi-\sin\frac{i.360}{n}\sin 2\phi,$$

$$\cos\left(\frac{i.360}{n}-2\phi\right) = \cos\frac{i.360}{n}\cos 2\phi+\sin\frac{i.360}{n}\sin 2\phi.$$

In terms of the theory of matrices the results obtained in connexion with forms deriving from symmetry with a unique axis may be summarized as follows:

Square N-rowed matrix for holohedral classes with a unique di-n-gonal axis.

$$||g_{ik}||_N = \left\| \begin{matrix} \text{I} & \text{II} & \text{III} & \text{IV} \\ \text{II}' & \text{I} & \text{IV}' & \text{III} \\ \text{III} & \text{IV} & \text{I} & \text{II} \\ \text{IV}' & \text{III} & \text{II}' & \text{I} \end{matrix} \right\| \qquad N = 4n$$

with square n-rowed submatrices as constituents:

$$\begin{aligned}
\text{I} &= \sin^2\rho||a_{ik}||_n + \cos^2\rho||c_{ik}||_n \\
\text{II} &= \sin^2\rho||b_{ik}||_n + \cos^2\rho||c_{ik}||_n \\
\text{II}' &= \sin^2\rho||b_{ki}||_n + \cos^2\rho||c_{ik}||_n \\
\text{III} &= \sin^2\rho||b_{ik}||_n - \cos^2\rho||c_{ik}||_n \\
\text{IV} &= \sin^2\rho||b_{ik}||_n - \cos^2\rho||c_{ik}||_n \\
\text{IV}' &= \sin^2\rho||b_{ki}||_n - \cos^2\rho||c_{ik}||_n \\
a_{ik} &= \cos\left[(k-i)\,\frac{2\pi}{n}\right];\ c_{ik} = 1 \\
b_{ik} &= \cos\left[(k-i)\frac{2\pi}{n} \pm 2\phi\right] \quad \text{with } + \text{ for } k \geqslant i,
\end{aligned}$$

$||b_{ki}||_n$ is the transposed form of $||b_{ik}||_n$. Therefore II′ and IV′ are the transposed forms of II and IV.

Because $g_{ii} = 1$, the trace of $||g_{ik}||_N = N$.

The necessary and sufficient conditions have now been given which must be fulfilled by the cosine values of the angles between any plane and the other equivalent ones if the general matrix figure 1 is to possess the symmetry corresponding to forms with a unique axis. No distinction has been made between crystallographic and non-crystallographic forms. The results may briefly be stated as follows:

Equipoints as in the di-n-gonal bipyramid. The matrix contains the I-, II-, II′-, III-, IV-, IV′-squares. The sum of the constituents in each row or column is zero.

Equipoints as in the n-gonal bipyramid. The matrix contains two I- and two III-squares. The sum of the constituents in each row or column is zero.

Equipoints as in the n-gonal trapezohedron. The matrix contains two I-, one IV-, and one IV′-square. The sum of the constituents in each row or column is zero.

Equipoints as in the di-n-gonal pyramid. The matrix contains I-, II-, II′-, I-squares. The sum of the constituents in each row or column is $2n\cos^2\rho$. For the di-n-gonal prism $\cos\rho = 0$ and the sum of the constituents in each row or column again becomes zero.

Equipoints as in the n-gonal pyramid. The matrix contains only **I**. The number of different values is $n/2$ or $(n+1)/2$. The sum of the constituents in each row or column is $n \cos^2\rho$ and zero for the n-gonal prism. For matrices corresponding to scalenohedra and streptohedra (e.g. rhombohedra) see page 316.

T. Liebisch[1] considered the angles $1/1'$ and also $1/n'$ and $1/(1)$ as fundamentally important for di-n-gonal di-pyramids. They can be deduced[2] from our general formulae for any given n and as functions of ϕ and ρ. The same applies to all fundamental angles of forms deriving from symmetry with a unique axis. It is equally easy to determine the distances between equivalent points arranged around a central point. This is an important problem in the investigation of co-ordination patterns within crystal structures. For this purpose the distance of a point from the chief point of symmetry is taken as 1. If the lines between equivalent points and the chief point of symmetry comprise the angle ϵ, the square of the distance between the points is then $d_\epsilon^2 = 2 - 2\cos\epsilon$. For $\cos\epsilon$ we can substitute the α_i–, β_i–, a_i–, b_i–values. If in a general matrix containing the d^2–values, the sum of the cosines be zero (when α_i, β_i, a_i, b_i are the constituents of the rows and columns), then the matrix of the d^2-values must consist of numbers whose sum in each row or column of the subsquares is $2n$. This is true for all non-polar forms. For groups corresponding to the di-n-gonal di-pyramid the sums of the rows or columns containing d^2–values is $8n$.

[1] Theodor Liebisch, Geometrische Krystallographie. Léipzig, 1881, p. 223.

[2] The following formulae are convenient in calculations connected with forms deriving from symmetry with a unique axis.

$$\alpha_1 = \cos^2\rho + \sin^2\rho \cos\frac{2\pi}{n}; \qquad \beta_0 = \cos^2\rho + \sin^2\rho \cos 2\phi$$

$$(1+a_0)\cos 2\phi = 2\beta_0 + a_0 - 1 = 2b_0 - a_0 + 1$$

$$(1+a_0)\cos\frac{2\pi}{n} = 2\alpha_1 + a_0 - 1 = 2a_1 - a_0 + 1$$

$$(1+a_0)\cos\left(\frac{2\pi}{n} - 2\phi\right) = 2\beta_1 + a_0 - 1 = 2b_{\bar{1}} - a_0 + 1$$

$$(1+a_0)\cos\left(\frac{2\pi}{n} + 2\phi\right) = 2\beta_1 + a_0 - 1 = 2b_1 - a_0 + 1$$

It therefore follows that:

$$a_1 = \alpha_1 + a_0 - 1; \qquad b_{\bar{1}} = \beta_{\bar{1}} + a_0 - 1; \qquad b_0 = \beta_0 + a_0 - 1, \text{ etc.}$$

The ϕ angles and $\cos\frac{2\pi}{n}$ can each be calculated from two angular values. As

$$\cos\left(\frac{2\pi}{n} - 2\phi\right) + \cos\left(\frac{2\pi}{n} + 2\phi\right) = 2\cos\frac{2\pi}{n}\cos 2\phi,$$

further equations can be derived.

The characteristic values for $n = 6$ in both representations are given in the top rows of tables II and III.

TABLE II

Cosine values of the angles for $n = 6$.

I

	1	2	3
1	1	$\cos^2\rho+\frac{1}{2}\sin^2\rho$	$\cos^2\rho-\frac{1}{2}\sin^2\rho$
	4	5	6
1	$\cos^2\rho-\sin^2\rho$	$\cos^2\rho-\frac{1}{2}\sin^2\rho$	$\cos^2\rho+\frac{1}{2}\sin^2\rho$

II

	1′	2′	3′
1	$\cos^2\rho+\sin^2\rho\cos 2\phi$	$\cos^2\rho+\sin^2\rho\cos(60+2\phi)$	$\cos^2\rho+\sin^2\rho\cos(120+2\phi)$
	4′	5′	6′
1	$\cos^2\rho-\sin^2\rho\cos 2\phi$	$\cos^2\rho+\sin^2\rho\cos(120-2\phi)$	$\cos^2\rho+\sin^2\rho\cos(60-2\phi)$

III

	(1)	(2)	(3)
1	$\sin^2\rho-\cos^2\rho$	$-\cos^2\rho+\frac{1}{2}\sin^2\rho$	$-\cos^2\rho-\frac{1}{2}\sin^2\rho$
	(4)	(5)	(6)
1	-1	$-\cos^2\rho-\frac{1}{2}\sin^2\rho$	$-\cos^2\rho+\frac{1}{2}\sin^2\rho$

IV

	(1′)	(2′)	(3′)
1	$-\cos^2\rho+\sin^2\rho\cos 2\phi$	$-\cos^2\rho+\sin^2\rho\cos(60+2\phi)$	$-\cos^2\rho+\sin^2\rho\cos(120+2\phi)$
	(4′)	(5′)	(6′)
1	$-\cos^2\rho-\sin^2\rho\cos 2\phi$	$-\cos^2\rho+\sin^2\rho\cos(120-2\phi)$	$-\cos^2\rho+\sin^2\rho\cos(60-2\phi)$

TABLE III

d^2-values for $n = 6$ (squares of the distances of equivalent points) on sphere with unit radius.

I

	1	2	3	4	5	6
1	0	$\sin^2\rho$	$3\sin^2\rho$	$4\sin^2\rho$	$3\sin^2\rho$	$\sin^2\rho$

II

	1′	2′	3′
1	$2\sin^2\rho(1-\cos 2\phi)$	$2\sin^2\rho[1-\cos(60+2\phi)]$	$2\sin^2\rho[1-\cos(120+2\phi)]$
	4′	5′	6′
1	$2\sin^2\rho(1+\cos 2\phi)$	$2\sin^2\rho[1-\cos(120-2\phi)]$	$2\sin^2\rho[1-\cos(60-2\phi)]$

III

	(1)	(2)	(3)	(4)	(5)	(6)
1	$4\cos^2\rho$	$1+3\cos^2\rho$	$3+\cos^2\rho$	4	$3+\cos^2\rho$	$1+3\cos^2\rho$

IV

	(1′)	(2′)	(3′)
1	$4-2\sin^2\rho(1+\cos 2\phi)$	$4-2\sin^2\rho[1+\cos(60+2\phi)]$	$4-2\sin^2\rho[1+\cos(120+2\phi)]$
	(4′)	(5′)	(6′)
1	$4-2\sin^2\rho(1-\cos 2\phi)$	$4-2\sin^2\rho[1+\cos(120-2\phi)]$	$4-2\sin^2\rho[1+\cos(60-2\phi)]$

In the case of $n = 6$, simpler expressions can be obtained by making use of the relations

$$\cos(120+2\phi) = -\cos(60-2\phi); \qquad \cos(60+2\phi) = -\cos(120-2\phi).$$

Also, of course, $\cos^2\rho$ can be recalculated in terms of $\sin^2\rho$ and vice versa.

2. *Matrix representation for isometric symmetry*

Beside the point symmetries characterized by the matrix table I*a*, the following additional equivalent point symmetries now occur:

in cubic symmetry: $N = 4, 6, 8, 12, 24, 48$

in icosahedral symmetry: $N = 12, 20, 30, 60, 120$.

The special forms of the matrix can be deduced for these cases also, but here it is not proposed to go beyond a general discussion of the forms deriving from cubic symmetry.

In the case of the cubic 48-point group we restrict ourselves to giving the cosine values of the top row of the complete matrix out of which everything else follows. The crystallographer will easily follow the sequence used here if the indices of the planes are states whose angles with *hkl* correspond to the given cosines (table IV*a*). However, the

TABLE IV *a*

Top row of the subsquares I–IV for cubic 48-point group

Cosine of the angle included between $h\,k\,l$ and:													Number of different *z* values
I	$h\,k\,l$	$k\,l\,h$	$l\,h\,k$	$h\,\bar k\,\bar l$	$k\,\bar l\,\bar h$	$l\,\bar h\,\bar k$	$\bar h\,k\,\bar l$	$\bar k\,l\,\bar h$	$\bar l\,h\,\bar k$	$\bar h\,\bar k\,l$	$\bar k\,\bar l\,h$	$\bar l\,\bar h\,k$	
Cosine symbol	A_0	B_0	B_0	A_1	B_3	B_2	A_2	B_1	B_3	A_3	B_2	B_1	8
II	$\bar h\,\bar l\,\bar k$	$\bar l\,\bar k\,\bar h$	$\bar k\,\bar h\,\bar l$	$\bar h\,l\,k$	$\bar l\,k\,h$	$\bar k\,h\,l$	$h\,\bar l\,k$	$l\,\bar k\,h$	$k\,\bar h\,l$	$h\,l\,\bar k$	$l\,k\,\bar h$	$k\,h\,\bar l$	
Cosine symbol	C_1	C_2	C_3	D_1	E_2	E_3	E_1	D_2	E_3	E_1	E_2	D_3	9
III	$\bar h\,\bar k\,\bar l$	$\bar k\,\bar l\,\bar h$	$\bar l\,\bar h\,\bar k$	$\bar h\,k\,l$	$\bar k\,l\,h$	$\bar l\,h\,k$	$h\,\bar k\,l$	$k\,\bar l\,h$	$l\,\bar h\,k$	$h\,k\,\bar l$	$k\,l\,\bar h$	$l\,h\,\bar k$	
Cosine symbol	$-A_0$	$-B_0$	$-B_0$	$-A_1$	$-B_3$	$-B_2$	$-A_2$	$-B_1$	$-B_3$	$-A_3$	$-B_2$	$-B_1$	8
IV	$h\,l\,k$	$l\,k\,h$	$k\,h\,l$	$h\,\bar l\,\bar k$	$l\,\bar k\,\bar h$	$k\,\bar h\,\bar l$	$\bar h\,l\,\bar k$	$\bar l\,k\,\bar h$	$\bar k\,h\,\bar l$	$\bar h\,\bar l\,k$	$\bar l\,\bar k\,h$	$\bar k\,\bar h\,l$	
Cosine symbol	$-C_1$	$-C_2$	$-C_3$	$-D_1$	$-E_2$	$-E_3$	$-E_1$	$-D_2$	$-E_3$	$-E_1$	$-E_2$	$-D_3$	9

rules apply quite irrespectively of the rationality of the indices. I contains the tetartohedral equipoints which together with II produce the enantiomorphic, with III the paramorphic, and with IV the hemimorphic classes of symmetry. I to IV are required for cubic holohedral symmetry. Where the same numerical values with negative signs occur in different submatrices, the same letter preceded by a negative sign has been used for the cosine. It is apparent that the 48-point form possesses 17 different values, each appearing positive and negative. Thus there are 34 angular values in all. The matrix of the cubic holohedral 48-plane form (or point group) contains as submatrices all the matrices of the subgroups contained in the group and belonging to the cubic hyposyngony. The ϕ- and ρ- values of the plane of departure are, of course, those corresponding to the cubic setting. The composition of the top row in the matrix in each individual case can be taken from table IV *b* in which N, Z, and z of the general form or point group are given.

TABLE IV *b*. Values for the subgroups of O_h

		N	Z	z
O_h	I II III IV	48	34	17
O	I II	24	17	17
T_d	I IV	24	17	17
T_h	I III	24	16	8
T	I	12	8	8
D_{4h}	$\pm A_0\ A_1\ A_2\ A_3\ C_3\ D_3\ E_3$	16	14	7
D_4	$+A_0\ A_1\ A_2\ A_3\ C_3\ D_3\ E_3$	8	7	7
C_{4h}	$\pm A_0\ A_3\ E_3$	8	6	3
C_{4v}	$+A_0\ A_3\ E_3-A_1\ A_2\ C_3\ D_3$	8	7	7
D_{2d}	$+A_0\ A_1\ A_2\ A_3-C_3\ D_3\ E_3$	8	7	7
C_4	$+A_0\ A_3\ E_3$	4	3	3
S_4	$+A_0\ A_3-E_3$	4	3	3
D_{2h}	$\pm A_0\ A_1\ A_2\ A_3$	8	8	4
D_2	$+A_0\ A_1\ A_2\ A_3$	4	4	4
C_{2v}	$+A_0\ A_3-A_1\ A_2$	4	4	4
C_{2h}	$\pm A_0\ A_2$	4	4	2
C_2	$+A_0\ A_2$	2	2	2
C_s	$+A_0-A_2$	2	2	2
C_i	$\pm A_0$	2	2	1
C_1	$+A_0$	1	1	1
D_{3d}	$\pm A_0\ B_0\ C_1\ C_2\ C_3$	12	10	5
D_3	$+A_0\ B_0\ C_1\ C_2\ C_3$	6	5	5
C_{3v}	$+A_0\ B_0-C_1\ C_2\ C_3$	6	5	5
C_{3i}	$\pm A_0\ B_0$	6	4	2
C_3	$+A_0\ B_0$	3	2	2

The composition of the cosine values symbolized by A_i, B_i, C_i, D_i, E_i is given in table V which consists of four parts. Of these section (*a*) contains a number of computation values which can usefully be derived at the outset of and used during a calculation. Section (*b*) then shows in what manner the 17 cosine values derive from these preliminary ones. The formulae are independent of the law of rationality, but section (*c*) of the table shows the connexions between h, k, and l and the product of two indices on the one hand, and the ϕ-, ρ-values on the other in cases where the law is applicable. In substituting for other planes due care must be given to the signs and sequences of the indices.

The cosines of the angles between (hkl) and the other equivalent rational planes can, as shown by section (*c*) of table V, always be written in the form $N/(h^2+k^2+l^2)$ in which N can assume the various values given in section (*d*) of table V.

TABLE V. (Explaining Table IV)

(*a*) *Computation values*

$p = \cos^2\rho$	$r = \sin^2\rho \sin^2\phi$	$t = \sin^2\rho \sin 2\phi$	$v = \sin 2\rho \sin \phi$
$q = \sin^2\rho \cos^2\phi$	$s = \sin^2\rho \cos 2\phi$	$u = \sin 2\rho \cos \phi$	$w = \cos 2\rho = \cos^2\rho - \sin^2\rho$

(*b*) *Formation of the cosine values*

$A_0 = 1$	$B_0 = \frac{1}{2}(t+u+v)$	$C_1 = -r-u$	$D_3 = -p+t$
$A_1 = -p-s$	$B_1 = \frac{1}{2}(-t+u-v)$	$C_2 = -q-v$	$E_1 = r$
$A_2 = -p+s$	$B_2 = \frac{1}{2}(-t-u+v)$	$C_3 = -p-t$	$E_2 = q$
$A_3 = w$	$B_3 = \frac{1}{2}(t-u-v)$	$D_1 = -r+u$	$E_3 = p$
		$D_2 = -q+v$	

(*c*) *Indices expressed in* ρ *and* ϕ

$h^2 = \sin^2\rho \sin^2\phi\ (h^2+k^2+l^2)$	$kl = \frac{1}{2}\sin 2\rho \cos\phi\ (h^2+k^2+l^2)$
$k^2 = \sin^2\rho \cos^2\phi\ (h^2+k^2+l^2)$	$lh = \frac{1}{2}\sin 2\rho \sin\phi\ (h^2+k^2+l^2)$
$l^2 = \cos^2\rho\ (h^2+k^2+l^2)$	$hk = \frac{1}{2}\sin^2\rho \sin 2\phi\ (h^2+k^2+l^2)$

(*d*) *Calculations of the cosine values for planes with rational indices. Value of*

N in $\dfrac{N}{h^2+k^2+l^2}$	$N(B_0) = kl+lh+hk$*	$N(C_1) = -h^2-2kl$	$N(D_3) = -l^2+2hk$
	$N(B_1) = kl-lh-hk$*	$N(C_2) = -k^2-2lh$	$N(E_1) = h^2$*
$N(A_0) = h^2+k^2+l^2$	$N(B_2) = -kl+lh-hk$*	$N(C_3) = -l^2-2hk$	$N(E_2) = k^2$*
$N(A_1) = h^2-k^2-l^2$	$N(B_3) = -kl-lh+hk$*	$N(D_1) = -h^2+2kl$	$N(E_3) = l^2$*
$N(A_2) = -h^2+k^2-l^2$		$N(D_2) = -k^2+2lh$	
$N(A_3) = -h^2-k^2+l^2$			

* Can be formed in two different ways.

For the transitional and special forms of the cubic system new conditions arise. They appear whenever ρ or ϕ or both assume special values. Table VI contains all the necessary data. Of course the matrices corresponding to cases with $N < 48$ (e.g. 6, 12, 4, 8, 24) are correspondingly smaller. The complete matrix representation is only required for the hexahedral, tetrahedral, octahedral, and rhombic-dodecahedral point groups in which the angular values are uniquely determined. Tables VII, VIII, and IX give these matrices in the order selected for the 48-point group.

A quite similar treatment can be devoted to the icosahedral group, but the comparative unimportance of the 20-, 30-, 60-, and 120-point groups does not warrant their discussion in this paper.

The regular pentagonal dodecahedron of I and I_h which can also appear as a non-crystallographical form in T and T_h, has as ρ-value 90° and as ϕ-values 31° 43′ or 180° −31° 43′. Five neighbouring planes form angles of 63° 26′ with each plane of the regular pentagonal dodecahedron. As

$$\cos 2\phi = \tfrac{1}{2}\sin 2\phi,\ A_1 = B_1 = B_2 = -0{\cdot}4473$$

and $$A_2 = B_0 = B_3 = 0{\cdot}4473 \quad \text{(see table VI, col. 4).}$$

TABLE VI. Special values for transitional and special forms in the cubic system.

$N = 6$ hexahedral	$N = 12$ rhombic-dodecahedral	$N = 4$ tetrahedral $N = 8$ octahedral	$N = 12$ pentagonal dodecahedral $N = 24$ tetrakishexahedral	$N = 12$ {deltoiddodecahedral, triakistetrahedral} $N = 24$ {triakisoctahedral, deltoidicositetrahedral}
$\rho = 0, \phi = 0$	$\rho = \pi/4, \phi = 0$	$\rho = 54° 44' 8'', \phi = \pi/4$	$\rho = \pi/2$, ϕ any value	ρ any value, $\phi = \pi/4$
$q = r = s = t = u = v = 0$ $p = w = 1$	$r = t = v = w = 0$ $p = q = s = \frac{1}{2}$ $u = 1$	$s = 0$ $p = q = r = \frac{1}{3}$ $t = u = v = \frac{2}{3}$ $w = -\frac{1}{3}$	$p = u = v = 0$ $w = -1$ $q = \cos^2\phi$ $r = \sin^2\phi$ $s = \cos 2\phi$ $t = \sin 2\phi$	$s = 0$ $p = \cos^2\rho$ $t = \sin^2\rho$ $q = r = \frac{1}{2}\sin^2\rho$ $w = \cos 2\rho$ $u = v = \frac{1}{2}\sqrt{2}\sin 2\rho$
$B_0 = B_1 = B_2 = B_3 = C_1 = C_2$ $= D_1 = D_2 = E_1 = E_2 = 0$ $A_0 = A_1 = A_2 = A_3 = 1$ $C_3 = D_3 = E_3 = -1$	$A_2 = A_3 = E_1 = 0$ $B_0 = B_1 = E_2 = E_3 = \frac{1}{2}$ $B_2 = B_3 = C_2 = C_3$ $= D_2 = D_3 = -\frac{1}{2}$ $A_0 = D_1 = 1$ $A_1 = C_1 = -1$	$D_1 = D_2 = D_3 = E_1$ $= E_2 = E_3 = \frac{1}{3}$ $A_1 = A_2 = A_3 = B_1$ $= B_2 = B_3 = -\frac{1}{3}$ $A_0 = B_0 = 1$ $C_1 = C_2 = C_3 = -1$	$E_3 = 0$ $A_0 = 1$ $A_3 = -1$ $E_2 = \cos^2\phi$ $C_2 = D_2 = -\cos^2\phi$ $E_1 = \sin^2\phi$ $C_1 = D_1 = -\sin^2\phi$ $A_2 = \cos 2\phi$ $A_1 = -\cos 2\phi$ $D_3 = \sin 2\phi$ $C_3 = -\sin 2\phi$ $B_0 = B_3 = \frac{1}{2}\sin 2\phi$ $B_1 = B_2 = -\frac{1}{2}\sin 2\phi$	$A_0 = 1$ $C_3 = -1$ $E_3 = \cos^2\rho$ $A_1 = A_2 = -\cos^2\rho$ $E_1 = E_2 = \frac{1}{2}\sin^2\rho$ $B_1 = B_2 = -\frac{1}{2}\sin^2\rho$ $A_3 = \cos 2\rho = \cos^2\rho - \sin^2\rho$ $D_3 = -\cos 2\rho = \sin^2\rho - \cos^2\rho$ $B_0 = \frac{1}{2}\sin^2\rho + \frac{1}{2}\sqrt{2}\sin 2\rho$ $C_1 = C_2 = -\frac{1}{2}\sin^2\rho - \frac{1}{2}\sqrt{2}\sin 2\rho$ $B_3 = \frac{1}{2}\sin^2\rho - \frac{1}{2}\sqrt{2}\sin 2\rho$ $D_1 = D_2 = -\frac{1}{2}\sin^2\rho + \frac{1}{2}\sqrt{2}\sin 2\rho$
$Z = 3$ $z = 2$ 5 or 14 equations	$Z = 5$ $z = 3$ 3 or 12 equations	$N = 4$: $Z = 2$, $z = 2$; $N = 8$: $Z = 4$, $z = 2$ 6 or 13 equations	$N = 12$: $Z = 6$, $z = 3$; $N = 24$: $Z = 13$, $z = 7$ 2 or 4 equations	$N = 12$: $Z = 6$, $z = 6$; $N = 24$: $Z = 12$, $z = 6$ 2 or 5 equations

TABLE VII. Hexahedral matrix. (Cosine values.)

	001	010	100	00$\bar{1}$	0$\bar{1}$0	$\bar{1}$00
	1	2	3	4	5	6
1	1	0	0	−1	0	0
2	0	1	0	0	−1	0
3	0	0	1	0	0	−1
4	−1	0	0	1	0	0
5	0	−1	0	0	1	0
6	0	0	−1	0	0	1

TABLE VIII. Tetrahedral and octahedral matrix. (Cosine values.)

		I				II				
		111	1$\bar{1}\bar{1}$	$\bar{1}$1$\bar{1}$	$\bar{1}\bar{1}$1	$\bar{1}\bar{1}\bar{1}$	$\bar{1}$11	1$\bar{1}$1	11$\bar{1}$	
		1	2	3	4	5	6	7	8	
I	1	1	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	−1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	II
	2	$-\frac{1}{3}$	1	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	−1	$\frac{1}{3}$	$\frac{1}{3}$	
	3	$-\frac{1}{3}$	$-\frac{1}{3}$	1	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	−1	$\frac{1}{3}$	
	4	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	−1	
II	5	−1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	I
	6	$\frac{1}{3}$	−1	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	1	$-\frac{1}{3}$	$-\frac{1}{3}$	
	7	$\frac{1}{3}$	$\frac{1}{3}$	−1	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	1	$-\frac{1}{3}$	
	8	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	−1	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	1	
		I				II				

I = Tetrahedron. I + II Octahedron

TABLE IX. Rhombic dodecahedral matrix. (Cosine values.)

	011	110	101	0$\bar{1}\bar{1}$	1$\bar{1}$0	10$\bar{1}$	01$\bar{1}$	$\bar{1}$10	$\bar{1}$0$\bar{1}$	0$\bar{1}$1	$\bar{1}\bar{1}$0	$\bar{1}$01
	1	2	3	4	5	6	7	8	9	10	11	12
1	1	$\frac{1}{2}$	$\frac{1}{2}$	−1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$
2	$\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	−1	$-\frac{1}{2}$
3	$\frac{1}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	−1	$\frac{1}{2}$	$-\frac{1}{2}$	0
4	−1	$-\frac{1}{2}$	$-\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$
5	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	−1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$
6	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	−1
7	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	−1	$-\frac{1}{2}$	$-\frac{1}{2}$
8	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	−1	$-\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$
9	$-\frac{1}{2}$	$-\frac{1}{2}$	−1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0
10	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	−1	$-\frac{1}{2}$	$-\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
11	$-\frac{1}{2}$	−1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$
12	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	−1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	1

There are in all only four different angles and the top line of the matrix in the usual arrangement therefore reads as follows:

A_0	B_0	B_0	A_1	B_3	B_2	A_2	B_1	B_3
1	0·4473	0·4473	−0·4473	0·4473	−0·4473	0·4473	−0·4473	0·4473

A_3	B_2	B_1
−1	−0·4473	−0·4473

$Z = 4$, $z = 2$. The number of equations governing the conditions is four or twelve.

We thus obtain an exact formulation of the laws to be obeyed by a normalized sequence of numbers in a square matrix if the matrix is to be an image of the properties of an equivalent complex within a symmetrical point group. This representation in matrix form is analogous to those called 'vector sets' and 'vector set matrices' by D. M. Wrinch[1] and M. J. Buerger[2] respectively. However, the matrices employed here are restricted to angular values or distances, i.e. to scalar quantities. The application of these methods to the symmetry of vector set matrices of symmetrical point groups presents no difficulties and would elaborate the remarks made by Buerger on this subject.

[1] D. M. Wrinch, Phil. Mag., 1939, vol. 27, p. 98.
[2] M. J. Buerger, Acta Cryst., Cambridge, 1950, vol. 3, p. 87.