The use of the gnomonic projection in the determination of the optical indicatrix of crystals.

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[Taken as read November 2, 1950.]

Introduction.

IN a previous paper[1] a description of a method was given which had been developed to determine the indicatrix of small crystals, both in orientation and magnitude, making use of the simple one-circle stage goniometer described by Bernal and Carlisle.[2]

As was explained there, when a crystal is viewed in parallel polarized light, between crossed nicols, the two observed extinction directions are the axes of the ellipse determined by the intersection of the indicatrix with a plane through its centre parallel to the plane of the microscope stage. For each position θ of the goniometer there are two extinction directions (ϕ_1 and ϕ_2) given by the readings on the microscope. These directions can be represented on a stereographic projection at suitable intervals of θ. It was shown there that the curves determined (on the indicatrix and on the stereographic projection) by these extinction directions during a complete rotation of the indicatrix about its fixed but arbitrary axis must contain the points representing the axes X, Y, and Z of the indicatrix. Furthermore, it was shown how to determine the actual orientation of these three axes and to measure the corresponding three principal refractive indices.

An attempt is made in the present paper to give the mathematical relations involved in the problem.

Theoretical treatment.

For the sake of simplicity, instead of rotating the indicatrix and studying its central sections parallel to the plane of the microscope stage, let us keep the indicatrix fixed and imagine a plane rotating about

[1] N. Joel, Min. Mag., 1950, vol. 29, p. 206.

[2] J. D. Bernal and C. H. Carlisle, Journ. Sci. Instruments, 1947, vol. 24, p. 107.

an axis which is parallel to the axis of the stage goniometer and goes through the centre of indicatrix. Moreover, let us imagine the indicatrix fixed with its Z-axis vertical. The rotation axis of the stage goniometer will no longer be horizontal, but will have an arbitrary—but fixed—orientation.

The equation of the indicatrix, with refractive indices equal to α, β, and γ along the X, Y, and Z axes, in cartesian co-ordinates, is

$$Ax^2+By^2+Cz^2 = 1, \tag{1}$$

where for simplicity (and throughout the whole paper), A, B, and C stand for α^{-2}, β^{-2}, and γ^{-2}, respectively.

Consider on the indicatrix the curves of constant refractive index. They are the intersections of surface (1) and the surfaces

$$x^2+y^2+z^2 = n^2, \tag{2}$$

where n is the refractive index and takes any value between α and γ. They are better represented in polar co-ordinates. If μ is the polar angle, measured from the Z-axis, and λ the azimuthal angle, measured from the X-axis towards the Y-axis, the equation is

$$\sin^2\mu = \frac{N-C}{A-C-(A-B)\sin^2\lambda}, \tag{3}$$

where N stands for n^{-2}. The rotation axis of the goniometer will be represented by its polar co-ordinates λ_0 and μ_0.

For the mathematical study of this particular problem the gnomonic projection is much more convenient than the stereographic projection, as the constant refractive index curves become conic sections and the rotating plane becomes a straight line rotating about a point.

If, for instance, the plane of the gnomonic projection is chosen perpendicular to the Z-axis of the indicatrix, at an arbitrary distance d from the XY plane, and its X and Y axes are chosen parallel to the X and Y axes of the indicatrix, the equation of the constant refractive index curves is:

$$(A-N)x^2+(B-N)y^2+(C-N)d^2 = 0. \tag{4}$$

The curves are ellipses for $C < N < B$ (that is $\beta < n < \gamma$); hyperbolae for $B < N < A$ ($\alpha < n < \beta$); and a pair of parallel straight lines,

$$x = \pm\frac{B-C}{A-B}d,$$

for $N = B$ ($n = \beta$), which are the projections of the two circular sections of the indicatrix (fig. 1).[1]

Consider, in fig. 1, any straight line. It represents a given section of the indicatrix and is a tangent to two of the constant refractive index curves (as will be shown below). The two contact points obviously represent the points of maximum and minimum refractive index along

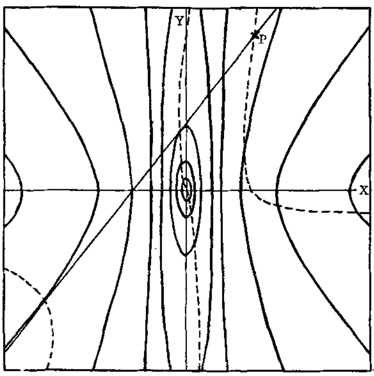

FIG. 1.

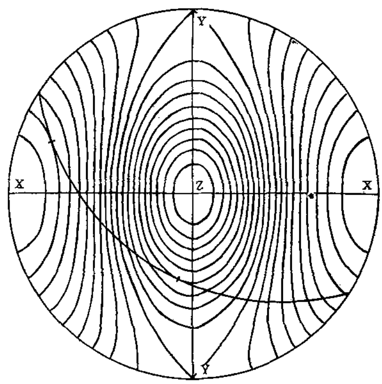

FIG. 2.

FIG. 1. The constant refractive index curves in the gnomonic projection for arbitrarily chosen values of α, β, and γ. The refractive index decreases from γ to α along the X-axis, and from γ to β along the Y-axis. Point P represents the goniometer axis, and the line through P, a section of the crystal. The dotted curves are the vibration-direction curves (or extinction-direction curves) for the given position of P.

FIG. 2. Stereographic projection of the curves of constant refractive index for a crystal with $A-B = B-C$, that is $2V = 90°$. The curves are drawn at regular intervals of N. Sections of the crystal are represented by great circles.

that line and therefore represent the two vibration directions in that given section. (It is easy to show that they are always 90° apart.) It follows that if a line rotates about a fixed point in the gnomonic projection (which corresponds to a plane rotating about the stage goniometer axis), the locus of the contact points will be the extinction-direction curve (or vibration-direction curve) mentioned in the first paper. Naturally the same holds for the stereographic projection, where it is a great circle that rotates about a fixed point.

[1] It should be noted that whatever the direction chosen for the plane of the gnomonic projection, the constant refractive index curves will always be conic sections. Their equations may be completely different, but the problem remains essentially the same.

We shall represent the rotation axis by the point $P(x_0, y_0)$, where $x_0 = d\tan\mu_0\cos\lambda_0$; $y_0 = d\tan\mu_0\sin\lambda_0$.

The tangents drawn from the point P to one of the curves (4) have contact points (x_1, y_1) and (x_2, y_2) given by

$$\left.\begin{aligned}(A-N)x^2+(B-N)y^2+(C-N)d^2 &= 0\\ (A-N)x_0x+(B-N)y_0y+(C-N)d^2 &= 0\end{aligned}\right\}. \tag{5}$$

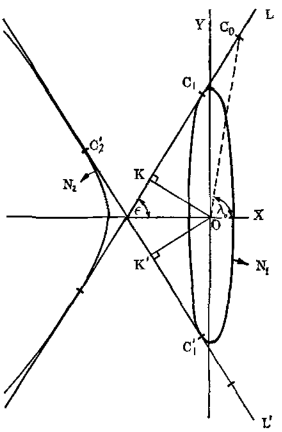

FIG. 3.

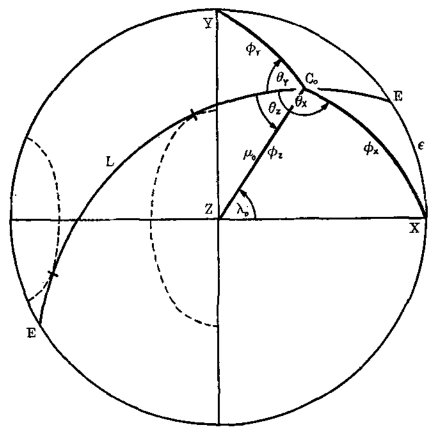

FIG. 4.

FIG. 3. Graphical construction (in gnomonic projection) to determine the orientation of the goniometer axis (C_0) with respect to the indicatrix.

FIG. 4. Diagram illustrating the change of the plane of the stereographic projection from the position normal to the Z-axis of the indicatrix to a plane parallel to the microscope stage. The polar angles ϕ are measured from the goniometer axis (C_0), and the azimuthal angles θ are measured from the plane of the microscope stage (great circle L).

When the line rotates about P, the locus of those contact points is obtained by eliminating N from the two equations (5)

$$(A-B)x_0xy^2-(A-B)y_0x^2y-(B-C)d^2y^2-(A-C)d^2x^2 \\ +(B-C)d^2y_0y+(A-C)d^2x_0x = 0, \tag{6}$$

or

$$(A-B)(x_0y-y_0x)xy+d^2(B-C)(y_0-y)y+d^2(A-C)(x_0-x)x = 0.$$

In fig. 1 the curve (6) corresponding to the given position of P is shown.

A line $y-y_0 = m(x-x_0)$ is a tangent to two of the constant refractive index curves (one ellipse and one hyperbola); their two N-values are the solutions of

$$pN^2-qN+r = 0, \tag{7}$$

where

$$p = d^2+m^2d^2+(y_0-mx_0)^2,$$

$$q = d^2(A+C)+m^2d^2(B+C)+(y_0-mx_0)^2(A+B),$$

$$r = d^2AC+m^2d^2BC+(y_0-mx_0)^2AB,$$

and they correspond to the two refractive indices of that particular section of the indicatrix.

Conversely, if A, B, C, N_1, and N_2 are given, it is possible to determine the equations of the common tangents to the two curves, and the co-ordinates of the contact points.

Practical application.

The above considerations, apart from explaining the behaviour of the indicatrix in the procedure of the method given in the previous paper, suggest, amongst other possibilities, the following application.

In order to find the optical orientation of a crystal of known optical constants, the answer may be obtained by following the method described in the previous paper, with small variations.

But, if for any reason (the crystal's shape, for instance) the crystal cannot be turned freely in the liquid, the following method is suggested:

(*a*) Mount the crystal on the goniometer.

(*b*) Fix the stage goniometer in a convenient position θ_0.

(*c*) Note the reading ϕ_0 on the microscope when the light transmitted by the polarizer vibrates parallel to the axis of the stage goniometer, and the ϕ_1 and ϕ_2 readings for the crystal in extinction position.

(*d*) Measure the two refractive indices of that given section and note whether they increase or decrease when the stage goniometer is turned slightly in an arbitrary sense.

(*e*) Choosing a convenient scale, draw the constant refractive index curves $(A-N)x^2+(B-N)y^2+(C-N)d^2 = 0$ for the two measured refractive indices,[1] and, to those two curves, draw the pair of common

[1] If many sections of the same crystal have to be studied, it is worth while to start drawing a complete chart for the given values A, B, and C at suitable intervals of N, either in the stereographic or in the gnomonic projection. If many sections of different crystals have to be examined, one single chart (in gnomonic projection) drawn for any arbitrary values of A, B, and C (for instance $A-B = B-C$, that is $2V = 90°$) may be used by altering the scale of the co-ordinates and choosing the

tangents that are not parallel to each other (fig. 3). Alternatively, instead of drawing the two conics, given A, B, C, N_1, and N_2, the co-ordinates of the four contact points may be calculated.

(f) From the measured refractive indices and the values of ϕ_0, ϕ_1 and ϕ_2, ϕ_1 may be associated to the contact points C_1 and C_1', ϕ_2 to C_2 and C_2', and it is possible to decide in which one of the three regions of each tangent line lie the points C_0 and C_0' (the eventual rotation axes) associated to ϕ_0.

(g) If a gnomonic net is available, the actual position of C_0 and C_0' will be easy to find. In the absence of a net, draw to the line $C_1 C_2$ the perpendicular OK from the origin, and

$$\overline{KC_0} = \sqrt{(d^2+\overline{OK}^2)}.\tan[\alpha+(\phi_0-\phi_1)],$$

where $$\alpha = \tan^{-1}[\overline{KC_1}/\sqrt{(d^2+\overline{OK}^2)}],$$

and, of course, $$\overline{K'C_0'} = \overline{KC_0}.$$

(h) To decide which of the two tangent lines actually represents the observed section of the crystal (which is parallel to the plane of the microscope stage), rotate the two tangent lines about C_0 and C_0' respectively, in the opposite sense to the rotation of the goniometer in (d). The one which gives the same variations as in (d) represents the section of the crystal and also the plane of the microscope stage. Let us suppose it is L; then C_0 (x_0, y_0) will be the goniometer axis.

The orientations of the microscope stage and the goniometer axis with respect to the indicatrix have been found. The polar co-ordinates of the axis are: $\tan\lambda_0 = y_0/x_0$; $\tan\mu_0 = \sqrt{(x_0^2+y_0^2)}/d$.

(i) The plane of projection, which for simplicity had been chosen perpendicular to the Z-axis of the indicatrix, can now be changed to a plane parallel to the microscope stage and the indicatrix referred to it. The three axes of the indicatrix will then have co-ordinates ϕ and θ, where ϕ is the polar angle measured from the goniometer axis and θ the azimuthal angle measured from the plane of the microscope stage. If a gnomonic net is available, either the rotation may be performed

proper N-values. The N-values and the factors (p and q) by which the scales have to be altered can be easily found from equations of the type:

$$\frac{p^2(A-N_1)}{A'-N_1'} = \frac{q^2(B-N_1)}{B'-N_1'} = \frac{C-N_1}{C'-N_1'},$$

and two similar equations with N_2 and N_2'. But if only a small number of sections are examined it is still more practical to draw only the pair of curves for each case.

by translating the line L to infinity, or the six angular co-ordinates determined with the help of the net.

If no gnomonic net is available it is also possible to make the change by inserting λ_0, μ_0, and ϵ (the angle between L and the X-axis) in a stereographic projection and, either performing the rotation by bringing the great circle L in coincidence with the primitive with the help of a stereographic net, or by directly reading the six co-ordinates on the net (fig. 4).

Alternatively, it is possible to calculate the ϕ and θ of the three axes. They are:

$$\phi_x = \cos^{-1}[\sin\mu_0 . \cos\lambda_0], \quad \theta_x = \theta_z + \cot^{-1}[-\cos\mu_0 . \cot\lambda_0],$$
$$\phi_y = \cos^{-1}[\sin\mu_0 . \sin\lambda_0], \quad \theta_y = \theta_z + \cot^{-1}[\cos\mu_0 . \tan\lambda_0],$$
$$\phi_z = \mu_0. \qquad \theta_z = \cot^{-1}[\cos\mu_0 . \cot(\lambda_0 - \epsilon)].$$

The problem can also be solved using only stereographic projections. The two curves to be plotted belong to the family of curves (equation 3), and with a stereographic net it is possible to find the great circle which touches both (fig. 4). The position of C_0 is then easy to find, and the change of primitive is performed as explained.

Acknowledgements.—I am very much indebted to Professor J. D. Bernal for guidance during the course of this work, and particularly for suggesting the use of the gnomonic projection. I also wish to thank Dr. G. Parry for help and advice on the presentation of the paper.