Determination of 2V from one extinction curve and its related n_0 curve

By N. Joel, M.Sc., Ph.D., A.Inst.P.

Centro de Investigaciones de Cristalografía, Instituto de Física y Matemáticas, Universidad de Chile, Casilla 2777, Santiago, Chile

[Taken as read 28 March 1963]

Summary. From the extinction curve of a biaxial crystal mounted on a spindle stage (or one-axis stage goniometer), the related n_0 curve can be derived and drawn on the stereogram. The n_0 curve is the equivibration curve—or constant-refractive-index curve—that goes through the projection P_0 of the spindle-stage axis. After the principal diameters (maximum 2η and minimum 2ξ) of the n_0 curve have been measured, the angle 2V may be calculated by means of the formula $\cos V = \sin \xi / \sin \eta$. This method can also be used for determining—or refining—the directions of the principal axes **α**, **β**, **γ**.

The use of extinction curve methods for determining the orientation of the optical indicatrix of small crystals has been described by several authors (Joel, 1950 and 1951; Joel and Garaycochea, 1957; Joel and Muir, 1958*a*, *b*; Wilcox, 1959; Fisher, 1962).

Wilcox (1959, 1960) has given a graphical procedure by which it is possible to determine, by successive trials, the value of 2V and the directions of the optic axes of a biaxial crystal mounted on a spindle stage, using an experimental extinction curve. This procedure requires that the directions of the principal axes[1] **α**, **β**, **γ** of the indicatrix be determined first, also from the extinction curve; but of the pair **α**, **γ**, one need not know which is **α** and which is **γ**.

Tocher (1962) has developed an interesting method for locating the directions of the two optic axes using extinction (or vibration) directions, thus proving that Joel and Muir (1958*a*, p. 876) were rather pessimistic when they thought that 'the positions of the optic axes cannot be located directly by the extinction curve method'. Tocher's method is

[1] The principal axes of the indicatrix are, of course, vectors, having both direction and magnitude. Many authors employ separate symbols: X, Y, Z for the directions, and α, β, γ (or N_α, N_β, N_γ, or N_X, N_Y, N_Z, or ...) for the magnitudes of the vectors **α**, **β**, **γ**. The Mineral Data Commission of the International Mineralogical Association has recommended that α, β, γ be used both for the magnitudes and for the directions of the axes, the precise connotation being given by the context; bold-face type (**α**, **β**, **γ**) will be used when an emphasis on the vector status is appropriate—(*Ed.*).

a graphical construction based on the Biot–Fresnel theorem, and it requires the optic axial plane **αγ** to be determined first: this he does from the extinction curve.

Garaycochea and Wittke (1963) have recently deduced a few simple formulae from which the angle 2V may be calculated after having determined from the extinction curve the directions α, β, γ. As their formulae require only the values of a few angular distances, which are read off the projection with a stereographic net, they are able to avoid making further graphical constructions for the determination of 2V once the extinction curve has been drawn and the α, β, γ axes have been located.

As the errors in the location of one or more of the three principal axes may affect the determination of the optic axes, it was thought of interest to minimize the errors in the former, or if possible, to devise a procedure by which the optic axes—or at least the value of 2V—could be determined directly from the extinction curve of a biaxial crystal mounted on a spindle stage (or one-axis stage goniometer) without having to locate first α, β, γ. This turned out to be possible and even quite simple.

The present paper consists of five sections and an appendix; in the first section the relation between the extinction curve and its n_0 curve—the equivibration curve through P_0—is explained. The second section deals with the procedure for obtaining the n_0 curve given the extinction curve. The third section gives an alternative technique for locating the three principal axes of the ellipsoid using the n_0 curve; this is as simple as the one first developed by Joel and Garaycochea (1957), it is capable of refinement and gives a better check of the accuracy achieved, and it makes easier the identification of the β axis. Furthermore, no 'ghost' axes are involved.[1] In the fourth section a very simple formula—related to the n_0 curve defined in the first section—is given, with which it is possible to calculate the value of the optic axial angle 2V without having to determine first the orientation of the ellipsoid, and without measuring refractive indices. An example and some discussion are given in the fifth section, while the mathematics of the problem are summarized in the appendix.

On the other hand, using a different approach, which is in fact a generalization of Tocher's (1962) method, Joel (1964) has developed a graphical construction that leads to the direct determination of the two optic axes—and consequently the three principal axes—from only

[1] It should be emphasized, however, that the ghost triangle has never, so far, caused any trouble, in whatever way the projected extinction curves are drawn.

four measured extinctions. This construction is rather lengthy, and it is mainly of theoretical interest; but it may be useful in cases where only a few reliable extinctions can be measured and consequently not all of the extinction curve is available.

Relation between the extinction curve and the n_0 curve

Extinction curves have various geometrical properties and, among others, the following theorems hold:

Theorem 1. If in a stereographic projection P_0 represents the direction of the rotation axis of the spindle stage, and one considers great circles through P_0, then the extinction curve is the locus of the points P that are midway between consecutive intercepts of these great circles with the circular sections of the ellipsoid (Joel and Garaycochea, 1957).

Theorem 2. The extinction curve is the locus of the points of tangency between those great circles through P_0 and successive equivibration curves[1] (Joel, 1951).

These equivibration curves—the loci of vibration directions (not propagation directions) in the crystal that have the same refractive index n —are ellipses on a sphere (spheroconics) that close around the axes $\boldsymbol{\alpha}$ and $\bar{\boldsymbol{\alpha}}$ respectively[2] if $\alpha < n < \beta$, and around the axes $\boldsymbol{\gamma}$ and $\bar{\boldsymbol{\gamma}}$ if $\beta < n < \gamma$; if $n = \alpha$ (or γ) the curves reduce to a pair of points, $\boldsymbol{\alpha}$ and $\bar{\boldsymbol{\alpha}}$ (or $\boldsymbol{\gamma}$ and $\bar{\boldsymbol{\gamma}}$); and if $n = \beta$ they coincide with the two great circles that represent the circular sections of the ellipsoid.

Theorem 3. The extinction curve is the locus of the points P that are midway between consecutive intercepts of the great circles through P_0 with the equivibration curve that goes through P_0 (fig. 1). This particular equivibration curve we shall call the n_0 curve, and it is the locus of the points that represent vibration directions in the crystal for which the associated wave has a refractive index equal to n_0.

In the next section it will be explained how the n_0 curve may be obtained from the experimental extinction curve for the purpose of locating the points α, β, and γ or calculating the angle 2V. An analytical proof of Theorem 3 is given in the appendix; but one can also approach the problem in this way: a diametral section of the ellipsoid through P_0 is an ellipse as shown in fig. 2. The vibration directions (extinction directions) OP and OP' are parallel to the two principal axes of this ellipse. It follows from symmetry considerations that if we choose Q such that the lengths OQ and OP_0 are equal, then the angles $\widehat{POP_0}$ and $\widehat{POQ}$ are equal too (and

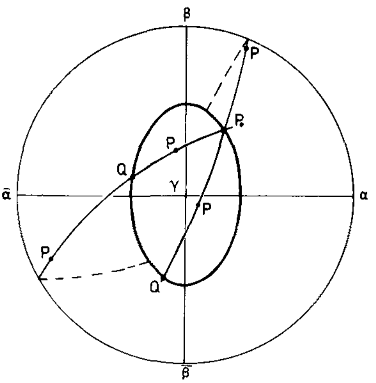
[1] The name *equivibration curve* will be adopted following Wright (1923) and Phemister (1954) as it is less ambiguous than that of *constant refractive index curve* used by Joel (1951).

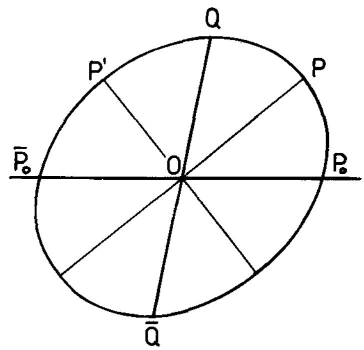
[2] Where appropriate, opposite ends of a diameter of the indicatrix are distinguished as $\alpha, \bar{\alpha}, P, \bar{P}$, &c.

similarly, the angles $\widehat{QOP'}$ and $\bar{P}_0\widehat{OP'}$ are equal). But P_0, $\bar{Q}$, $\bar{P}_0$, and Q belong to the equivibration curve that goes through P_0, or in other words, the n_0 curve; and P and P' belong to the extinction curve. Theorem 3 is thus proved.

Graphical construction of the n_0 curve given the extinction curve

To construct the n_0 curve once the extinction curve has been drawn is now quite simple (fig. 3). With the help of a stereographic net points such as Q are determined on great circles through P_0 so that the polar curve (or the equatorial curve, as the case may be) bisects P_0Q.





FIGS. 1 and 2. Fig. 1 (left). The n_0 curve (equivibration curve through P_0) in stereographic projection; points P halfway between consecutive intercepts of the great circles through P_0 with the n_0 curve are points of the extinction curve. Fig. 2 (right). Diametral section of the ellipsoid through P_0. OP and OP': vibration directions; P_0, Q, $\bar{P}_0$ and $\bar{Q}$: points of the n_0 curve.

For the purpose of drawing the n_0 curve, it is quite helpful, but not necessary, to proceed as follows: when plotting on the stereogram the pairs of points P and P' of the extinction curve (determined by the extinction angles $P_0P = \theta$, $P_0P' = 90°+\theta$), the point at an angular distance 2θ from P_0 (or $180°+2\theta$, as the case may be) should be plotted as well on the same great circle, as this is a point Q of the n_0 curve. Indeed, as was pointed out by Wittke (1962), it can be seen that in this way one could actually plot directly the n_0 curve without going through the intermediary stage of drawing the extinction curve. And, as is shown in the present paper, the three principal axes and the two optic axes, which can be derived from the extinction curve, can also be derived directly from the n_0 curve; in this, and possibly also in other aspects (though certainly not all), the n_0 curve can be as useful as the extinction curve. However, it seems more convenient to draw both the extinction curve and the n_0 curve.[1]

[1] There is no need to discuss uniaxial crystals in this paper, as in these the optic axis can be determined very easily from an extinction curve: the equatorial curve is a great circle and the optic axis is its pole (Joel, 1950; Joel and Muir 1958*a*). The optic axis lies on the polar curve. The n_0 curves, which can be obtained as explained above, are circles centred around the optic axis.

In figs. 3, 4, 5, and 6 of this paper the point P_0 has been chosen in the centre of the projection following the suggestion by Tocher (1962) and Fisher (1962). But there is no fundamental difference between the extinction curves plotted with P_0 in the centre and those with P_0 on the projection circle, or indeed anywhere. Of course, some graphical constructions become simpler in the first case and others in the second. (See also Fisher, 1962, p. 660.)

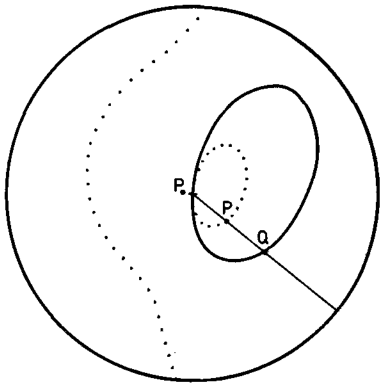


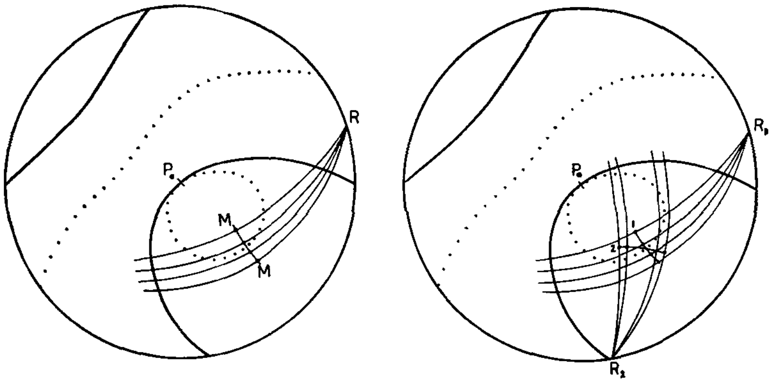
FIG. 3. Graphical construction of the n_0 curve (full line), given the extinction curve (dotted line): on great circles through P_0 the points Q are marked such that $PQ = P_0P$. In figs. 3 to 7 the spindle axis P_0 is in the centre of the stereographic projection.

Location of the principal axes by means of the n_0 curve

One of the principal axes, α or γ, has its projection on the polar curve; furthermore, this point is the centre of the n_0 curve that was obtained by means of the procedure given in the last section. The problem of locating the projection of this axis is therefore the problem of finding on the polar curve the point that is the centre of the n_0 curve. This can be done as shown in fig. 4. Any point R of the primitive circle is selected and, with the help of the stereographic net, the points that are half-way between the intercepts of the n_0 curve with the great circles through R are marked. The line joining these points cuts the polar curve at the desired point γ (or α).

The accuracy in the location of γ (or α) can be increased by repeating the outlined operation for several points R_1, R_2, etc. on the primitive circle (fig. 5). Actually, any point in the projection (except the point P_0 itself) can be taken as R, but the construction is simpler if it is taken on the primitive. Furthermore, the locus of the midpoints need only be drawn in the neighbourhood of the polar curve; and the points R may be selected so as to obtain intersections at favourable angles, and also so that the points of the n_0 curve that are used will correspond to the sharper experimental extinction positions.

Next the other two axes must be located. It is clear that the simplest way of achieving this is to draw the great circle perpendicular to the recently determined axis γ (or α). This great circle will intersect the equatorial curve in general in three points and two of these will be mutually perpendicular; these are the other two axes β and α (or γ).



FIGS. 4 and 5. Fig. 4 (left). Determination of the centre of the n_0 curve (full line): great circles through any point R intersect the n_0 curve, and the midpoints between these intersections are marked; the locus of these midpoints is the line MM, and the centre of the n_0 curve, in this case the point γ, is found as the intersection of MM with the polar curve (dotted line). Fig. 5 (right). Procedure for refining the location of γ (or α) on the polar curve: the two loci for the midpoints, relative to R_1 and R_2, are shown (R_1 is the point R of fig. 4).

The accuracy in locating these two axes can be considerably increased if one applies to them—or at least to one of them—the same procedure as was used for obtaining the first of the three axes. The points R can be the same as before but they need not be. They are chosen so as to give intersections at favourable angles, using if possible the more accurate parts of the n_0 curve (see above); again the midpoints need be located only in the regions where the two axes have already been approximately found; fig. 6 shows an example.

It is now quite simple to decide which of the two axes on the equatorial curve is β. It has already been pointed out (Joel and Garaycochea, 1957) that the polar curve stretches more towards the β axis, but this is not always very sensitive. The n_0 curve, on the other hand, shows this effect clearly: its maximum diameter points directly to β and its minimum one points to α (or γ). This means that, in theory at least, the actual positions of β and α (or γ) could be determined as the intersections of the equatorial curve with the two symmetry planes of the n_0 curve, that is, with its maximum and minimum diameters; but this is generally less accurate than the above procedures.

The optic axial plane $\alpha\gamma$ has thus been located, but it is not possible to decide from the extinction curve alone which is α and which is γ; some additional measurements or observations (or some knowledge of

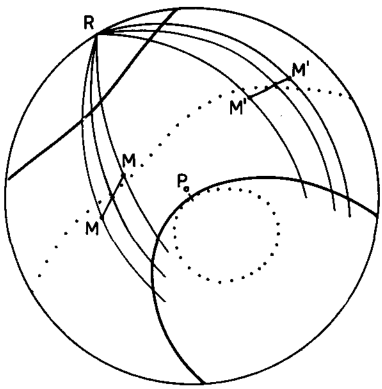


FIG. 6. Location of the two principal axes on the equatorial curve. The procedure is similar to that of figs. 4 and 5. MM determines β and $M'M'$ determines α.

the crystal that is being examined, for example, its optical sign) would be required to decide it. For instance, with a compensating plate one could find out whether the vibrations of the polar curve are slower or faster vibrations (have higher or lower refractive indices) than those of the equatorial curve, which would assign γ or α respectively to the polar curve. Or, for instance, one could compare the refractive index of the immersion liquid with those of the crystal along the equatorial curve: if the latter increase towards β, then γ is on the polar curve.

Having thus determined the orientation of the ellipsoid, it is possible to proceed to determine graphically the directions of the two optic axes—and hence the value of 2V—with either of the methods by Wilcox (1959, p. 1290; 1960) or Tocher (1962); or to calculate 2V with any of the formulae given by Garaycochea and Wittke (1963). However, a mathematical study of the n_0 curve led to an interesting method for determining 2V which has the advantage that its accuracy—which depends of course on the experimental errors in the extinction curve—is independent of the possible errors in the location of α, β, γ. This is so because it does not require the positions of α, β, γ to be known in advance, although the procedure is faster if one knows roughly the position of γ (or α) on the polar curve, which, as follows from what has been said here, is always easy. This new method for determining 2V, which requires no more than the measurement of the maximum and minimum diameters of the n_0 curve (it certainly does not require any refractive index measurements), is explained in the next section.

Determination of 2V *in terms of the two principal diameters of the* n_0 *curve*

An extinction curve is completely determined by the relative directions of the spindle-stage axis P_0 and the optic axes of the crystal (Joel and Garaycochea, 1957, page 400). It follows that the corresponding n_0 curve is also completely determined once the relative directions of the two optic axes and the spindle-stage axis are fixed. It was thought worthwhile, therefore, to search for a means of determining 2V once the n_0 curve was known. It should be remembered that the n_0 curve is obtained easily and quickly from the extinction curve, or even directly from the extinction readings on the microscope, without having to locate any of the axes α, β, γ.

A mathematical study of the n_0 curve showed that its two principal diameters (maximum 2η; minimum 2ξ) are related by a formula that involves the angle 2V. This formula is (the proof will be found in the appendix): $\cos V_\pi = \sin\xi/\sin\eta$ where V_π is the angle between one of the optic axes and that axis of the ellipsoid ($\pi = \alpha$ or γ) that lies on the polar curve. The angles ξ and η (fig. 7) are measured with the stereographic net, and the angle V_π is then calculated by the above formula.

This procedure does not require the points α, β, γ to be known in advance; but the measurements will be done much faster if they are known, even if only approximately. Indeed, if the principal axis (π) on the polar curve has been located, then one knows that the diameters to be sought must all pass through this point; and, as it is so easy to locate this point (see above), it would seem advisable to do so. Furthermore, if the other two principal axes, on the equatorial curve, have been located, then they determine the directions in which the two diameters of the spherical ellipse have to be measured. However, the value of each of these two diameters—though not always their direction—can be obtained with the same accuracy (but not so quickly) if the axes on the equatorial curve have not been previously located. It is in any case advisable to measure the diameters 2ξ and 2η over a reasonable range around the minimum and maximum respectively; this will give a more reliable result as it cancels out local imperfections of the n_0 curve. Also, one can measure diameters from points near the axis π and finally choose the one that gives the highest value of η and the lowest value of ξ, keeping in mind that the two principal diameters should bisect each other and be mutually perpendicular.

The accuracy of this determination of 2V is limited only—apart from drawing errors—by the quality of the extinction curve, and is unaffected by errors in the location of α, β, γ. This accuracy can be increased if—apart from the usual precautions such as having the crystal immersed in a liquid of appropriate refractive index—the points that correspond to sharp extinctions are given a special mark on the projection and more weight is given to these points when drawing the n_0 curve. If not only the value of 2V but also the directions of the optic axes A_1 and A_2 are required, then this value of 2V can be combined with the determination of the principal axes of the ellipsoid (see above).

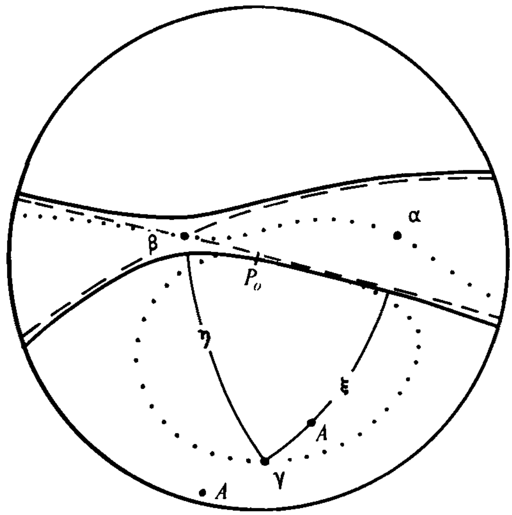


FIG. 7. Example of a determination of 2V with the n_0 curve: barite ($BaSO_4$). Dotted line, extinction curve; full line, n_0 curve. $\xi = 70°$, $\eta = 81{\cdot}5°$; calculated value of 2V, 36·3°. The circular sections (broken line) and the optic axes (A and A) have been added afterwards.

It will be remembered that without some additional observations, measurements, or comparisons we would not be able to know which of the two bisectrices is α and which is γ. But the optic axes are, nevertheless, determined without any ambiguity as the above formula gives unambiguously the angle V that each of them forms with the bisectrix that has been located on the polar curve (π); means of deciding whether this is a positive or a negative bisectrix have been indicated above.

Example and discussion

Fig. 7 shows an application of the n_0 method to a crystal of baryte, one of several examples that were tried. Working with red light, $\xi = 70°$, $\eta = 81{\cdot}5°$, and from the above formula 2V = 36·3°. According to Palache *et al.* (1951, p. 410) the angle is 36·5° for red light. With the same extinction curve 2V was also calculated by means of formulae (10), (17), and (20) of Garaycochea and Wittke (1963); each of these three formulae require different angular measurements on the projection. The results were 36·6°, 36·3°, and 36·6° respectively. In this example special efforts were made to obtain a consistently good extinction curve, which accounts for the good result. A few other extinction curves—with their n_0 curves—were obtained in various other settings of the same crystal, that is in different positions of the spindle axis relative to the optic axes of the crystal. These measurements were done in a routine fashion, and their n_0 curves led to values of 2V between 35·5° and 37·5°.

One can test the effect that errors $\Delta\xi$ and $\Delta\eta$ in ξ and η may have in a particular case on the calculated value of V_π, by recalculating V_π with $\xi+\Delta\xi$ and $\eta+\Delta\eta$ (the error ΔV is greatest when $\Delta\xi$ and $\Delta\eta$ have opposite signs). A general expression can be obtained, however, by differentiating the above equation, to give

$$\Delta V_\pi = (\cos\eta\sin\xi\,\Delta\eta - \sin\eta\cos\xi\,\Delta\xi)/\sin^2\eta\sin V_\pi.$$

One gets a useful formula if one assumes the errors $\Delta\xi$ and $\Delta\eta$ to have the same absolute value, say ϵ. The absolute value ϵ_v of the error in V_π will then be:

$$\epsilon_v = \epsilon\sin(\eta\mp\xi)/\sin^2\eta\sin V_\pi.$$

with the negative sign if $\Delta\xi$ and $\Delta\eta$ have the same sign.

From this formula it follows that when the n_0 curve has a diameter 2η approaching 180° and comes to be close to the circular sections (this happens when the spindle-stage axis P_0 is close to a circular section, that is, nearly perpendicular to one of the optic axes), then the maximum error ϵ_v in the angle V_π is approximately equal to ϵ; and that the error ϵ_v is greater when one is dealing with a small n_0 curve. In practice, however, some of the extinction settings in the former case will be unsharp (when one of the optic axes becomes nearly parallel to the microscope axis). It is therefore advisable, in general, unless special care is taken over the measurements, to make a compromise between a large and a small n_0 curve. By remounting the crystal on the spindle stage in a different orientation it is possible to obtain a different n_0 curve. Actually, this was achieved very conveniently, and also any direction in the crystal within a range of about 30° could be set parallel to the rotation axis of the instrument without unmounting the crystal, by means of a modified spindle stage designed by Villarroel (1964). On the other hand, in some cases the n_0 method can be used advantageously with the universal stage as well (Muir and Joel, 1964).

Conclusion

It would be difficult to make a general assessment of the relative merits of the various ways in which extinction curves have been used to date for determining the principal axes of the ellipsoid or the optic axes of a biaxial crystal mounted on a spindle stage. However, the main interest of the present method—in which the n_0 curves are used—derives from the fact that the angle 2V can be obtained quite simply and quickly without measuring refractive indices, without trial and error procedures, and without having to rely on the location of the axes α, β, γ or of the optic axial plane. Furthermore, an additional advantage of the n_0 method is that as the experimental n_0 curve should be an ellipse (a spherical ellipse), it becomes possible to check the quality of the experimental extinction curve by checking how much the n_0 curve conforms to a spherical ellipse. The first thing to do is to see if it actually has two symmetry planes (the principal diameters). Furthermore, one can not only check the average quality of the extinction curve, but also select its good parts from the bad ones.

Finally, it is important to emphasize that the various interesting properties of the extinction curves have the effect that in many aspects the different approaches complement each other, in as much as they provide opportunities for checking the experimental results and refining the graphical determinations.

Acknowledgements. I am grateful to Mrs. Isabel Garaycochea and Mr. Oscar Wittke for some comments and discussions, to Mr. Hugo Villarroel for assistance in the measurements and for his care and patience with the drawings, and to Dr. I. D. Muir for valuable suggestions concerning the presentation of this paper.

Appendix

Analytical proof of Theorem 3 (see fig. 1)

With the notation used by Joel and Garaycochea (1957), taken from Wilson (1943), we have—on a sphere of unit radius—that the equation of a point Q on the n_0 curve, or equivibration curve through P_0, is:

$$\mathbf{q}.\mathbf{\Phi}.\mathbf{q} = \mathbf{r}_0.\mathbf{\Phi}.\mathbf{r}_0 = n_0^{-2} \tag{1}$$

where $\mathbf{q}$ and $\mathbf{r}_0$ are the unit vectors that define the positions of the points Q and P_0; $\mathbf{\Phi}$ is the dyadic $\mathbf{\Phi} = \alpha^{-2}\mathbf{ii}+\beta^{-2}\mathbf{jj}+\gamma^{-2}\mathbf{kk}$; α, β, γ are the principal refractive indices of the crystal; and n_0 is the constant refractive index associated to all the vibration directions such as Q.

A point P, at the end of a unit vector $\mathbf{r}$, that is on the same great circle as P_0 and Q and at the same angular distance from both (fig. 1), is given by:

$$\left.\begin{aligned} \mathbf{r} &= \lambda\mathbf{r}_0+\mu\mathbf{q} \\ \mathbf{r}.\mathbf{q} &= \mathbf{r}.\mathbf{r}_0 \\ \mathbf{r}^2 = \mathbf{q}^2 &= \mathbf{r}_0^2 = 1 \end{aligned}\right\}. \tag{2}$$

From these equations the parameters λ and μ are determined, and the relation between $\mathbf{r}$ and $\mathbf{q}$ becomes:

$$\mathbf{q} = -\mathbf{r}_0+2(\mathbf{r}_0.\mathbf{r})\,\mathbf{r}. \tag{3}$$

This expression for $\mathbf{q}$ is now introduced in equation (1) because Q must be on the n_0 curve. The result is

$$(\mathbf{r}.\mathbf{\Phi}.\mathbf{r})(\mathbf{r}.\mathbf{r}_0) = \mathbf{r}.\mathbf{\Phi}.\mathbf{r}_0, \tag{4}$$

which is precisely the equation of the extinction curve (Joel and Garaycochea, 1957, page 405, formula 5).

The principal diameters 2ξ and 2η of the n_0 curve, and the formula for cos V.

The dyadic $\mathbf{\Phi}$ can be written in the form

$$\mathbf{\Phi} = B\mathbf{I}-\tfrac{1}{2}(A-C)(\mathbf{a}_1\mathbf{a}_2+\mathbf{a}_2\mathbf{a}_1), \tag{5}$$

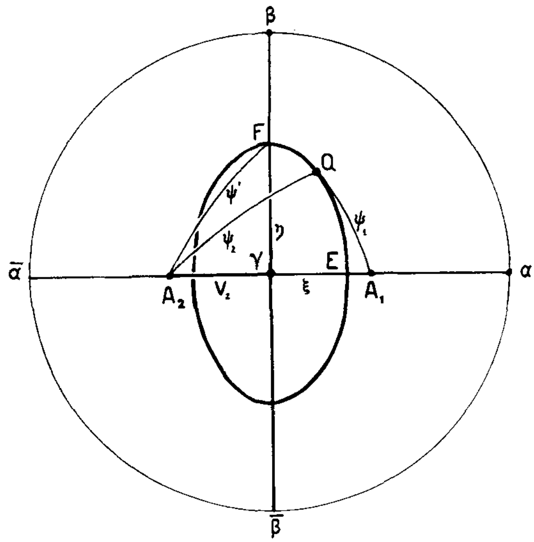


FIG. 8. Diagram to illustrate the derivation of the formula for cos V_π (the angle between either optic axis and the bisectrix that lies on the polar curve).

where $A = \alpha^{-2}$, $B = \beta^{-2}$, $C = \gamma^{-2}$; $\mathbf{a}_1$ and $\mathbf{a}_2$ are unit vectors parallel to the two optic axes of the crystal. The equation of the extinction curve becomes, by introducing (5) into (4) (Garaycochea and Wittke, 1963):

$$2(\mathbf{a}_1.\mathbf{r})(\mathbf{a}_2.\mathbf{r})(\mathbf{r}_0.\mathbf{r}) = (\mathbf{a}_1.\mathbf{r}_0)(\mathbf{a}_2.\mathbf{r})+(\mathbf{a}_2.\mathbf{r}_0)(\mathbf{a}_1.\mathbf{r}), \tag{6}$$

and the equation of the n_0 curve becomes, by combining (5) and (1),

$$(\mathbf{a}_1.\mathbf{q})(\mathbf{a}_2.\mathbf{q}) = (\mathbf{a}_1.\mathbf{r}_0)(\mathbf{a}_2.\mathbf{r}_0) = (B-N)/(A-C), \tag{7}$$

where $N = n_0^{-2}$.

Equation (7) can also be written

$$\cos\Psi_1 \cos\Psi_2 = (B-N)/(A-C), \tag{8}$$

where Ψ_1 and Ψ_2 are the angles between any vibration direction $\mathbf{q}$ of the n_0 curve and each of the optic axes $\mathbf{a}_1$ and $\mathbf{a}_2$ of the crystal.

The spherical ellipses defined by equations 7 or 8 close around γ if $B > N > C$, $(\beta < n_0 < \gamma)$, and around α if $A > N > B$, $(\alpha < n_0 < \beta)$. We will assume in the following analysis—without any loss in generality—that the principal axis on the polar curve is γ, and we are dealing, therefore, with a n_0 curve centred around γ. The constant $(B-N)/(A-C)$ in equations (7) and (8) is thus positive in this case.

It can be shown that the maximum diameter of this n_0 curve is in the $\beta\gamma$ plane and the minimum diameter is in the $\alpha\gamma$ plane. They can be obtained quite simply from equation (8). Let their lengths be 2η (maximum diameter) and 2ξ (minimum diameter). To get a formula for η let us consider point F in fig. 8. In the triangle $A_2\gamma F$:

$$\cos\Psi' = \cos V_\gamma \cos\eta \tag{9}$$

and the value of $\cos\Psi'$ can be obtained by setting $\Psi_1 = \Psi_2 = \Psi'$ in equation (8).

$$\cos\Psi'^2 = (B-N)/(A-C). \tag{10}$$

As the optic axial angle $2V_\gamma$ is known to be given by

$$\cos^2 V_\gamma = (B-C)/(A-C) \tag{11}$$

it follows that

$$\cos^2\eta = (B-N)/(B-C). \tag{12}$$

To get now a formula for ξ, let us consider point E in fig. 8; equation (8) takes for it the form

$$\cos(V_\gamma-\xi)\cos(V_\gamma+\xi) = (B-N)/(A-C) \tag{13}$$

and it follows that

$$\cos^2 V_\gamma+\cos^2\xi-1 = (B-N)/(A-C) \tag{14}$$

and that, combining this with (11)

$$\cos^2\xi = (A-N)/(A-C). \tag{15}$$

Equations (12) and (15) give the required expressions for the semi-diameters η and ξ, and combining them with equation (11), we can eliminate first N and then A, B, and C, and finally write an expression for V_γ in terms of ξ and η:

$$\cos V_\gamma = \sin\xi/\sin\eta. \tag{16}$$

If the principal axis of the indicatrix that lies in the polar curve happens to be α, we arrive at exactly the same expression for $\cos V_\alpha$, and we can therefore combine them and write:

$$\cos V_\pi = \sin\xi/\sin\eta \tag{17}$$

which is the formula used above for determining the optic axial angle 2V.

Equation (17) could also be proved by making use of the curve defined by the directions of the wavenormals associated with a given refractive index n_0. This curve is also an ellipse on the sphere of unit radius, its foci being precisely the points A_1 and A_2; and it can be shown that it is the reciprocal of the n_0 curve. The lengths of its two semi-diameters a and b are thus the complements of ξ and η, and they are related to V_π (the distance between one focus and the centre) by: $\cos a = \cos b \cos V_\pi$.

References

FISHER (D. J.), 1962. Amer. Min., vol. 47, p. 649.
GARAYCOCHEA (I.) and WITTKE (O.), 1963. Acta Cryst. (In press.)
JOEL (N.), 1950. Min. Mag., vol. 29, p. 206.
—— 1951. Ibid., vol. 29, p. 602.
—— and GARAYCOCHEA (I.), 1957. Acta Cryst., vol. 10, p. 399.
—— and MUIR (I. D.), 1958*a*. Min. Mag., vol. 31, p. 860.
—— —— 1958*b*. Ibid., p. 878.
—— 1964. Ibid., vol. 33, no. 264 (in press).
MUIR (I. D.) and JOEL (N.), 1964. (In preparation).

Palache (C.), Berman (H.), and Frondel (C.), 1951. Dana's System of Mineralogy, 7th edn, vol. 2.
Phemister (T. C.), 1954. Amer. Min., vol. 39, p. 172.
Tocher (F. E.), 1962. Min. Mag., vol. 33, p. 52.
Villarroel (H.) and Joel (N.), 1964. (In preparation).
Wilcox (R. E.), 1959. Amer. Min., vol. 44, p. 1272.
—— 1960. Bull. Geol. Soc. Amer., vol. 71, p. 2003.
Wilson (E. B.), 1943. Vector Analysis. New Haven (Yale University Press).
Wittke (O.), 1962. Personal communication.
Wright (F. E.), 1923. Journ. Opt. Soc. Amer., vol. 7, p. 779.