A stereographic construction for determining the optic axes of a biaxial crystal directly from a few extinction measurements

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Summary. It is possible to determine the directions of the optic axes of a biaxial crystal mounted on a spindle stage—and hence the angle $2V$ and the directions of α, β, γ—from only four measured extinction positions. The method is a generalization of the one proposed by Tocher (1962); but in the present procedure it is not necessary to determine first the optic axial plane.

WILCOX (1959, p. 1290; 1960) and Tocher (1962) have given stereographic constructions for determining, by means of an experimentally obtained extinction curve, the optic axes of a biaxial crystal mounted on a spindle stage. They require some previous knowledge of the orientation of the ellipsoid, which is also obtained from the extinction curve.

Wilcox determines first, from the experimental extinction curve (Joel, 1950; Joel and Garaycochea, 1957), the positions of α, β, and γ relative to the spindle axis P_0; and then draws theoretical extinction curves using different trial values of the optic axial angle $2V$. The correct value of $2V$ is the one that fits best the experimental extinction curve.

Tocher's method is a direct one: he locates first, also from the experimental extinction curve, the optic axial plane; and then locates on the latter the two points A_1 and A_2 (optic axes) using the Biot–Fresnel theorem for two wave-normals and their corresponding vibration directions: N_1, V_1, V_1' and N_2, V_2, V_2' respectively. A_1 and A_2 are determined by the condition that for both waves: $\angle A_1NV = \angle A_2NV$ (or $\angle A_1NV' = \angle A_2NV'$). This graphical construction is in fact quite simple and elegant: and as to its accuracy, Tocher himself says (1962, p. 56): 'A check on the accuracy of both the above graphical construction and the observations may now be made by ensuring that the optic axes so found are symmetrical with respect to any other wavenormal : vibration set.'

A further check is provided by the fact that if the points α and γ have been previously located on the optic axial plane, one of them should bisect the arc A_1A_2. If this does not happen, then the determination of A_1 and A_2 is repeated for slightly different positions of α, β, γ until a solution is obtained where α and γ do bisect the optic axial angle (Tocher, 1962, pp. 57, 58).

In order to make the accuracy of the location of A_1 and A_2 independent of possible errors in the location of the optic axial plane, it was considered of interest to try to generalize Tocher's stereographic method in the following way: only the directions of wave-normals (of course more than two) and the directions of the corresponding vibrations should be used, without assuming anything at all about the optic axial plane. Furthermore, it also became of interest to discuss how many experimentally recorded extinctions are actually necessary to determine A_1 and A_2.

It should be mentioned, before proceeding, that two further solutions, which are quite easy to apply, have recently been given for the problem of determining the angle $2V$ directly from an extinction curve. Garaycochea and Wittke (1964) have studied some further properties of the extinction curves and have derived some formulae each of which can be used for calculating the value of $2V$ from measurements made on the experimental extinction curve. And Joel (1963) has proposed another formula, also for calculating $2V$, making use of the n_0 curve (equivibration curve through P_0), which can be derived from the extinction curve. Both these procedures are quite simple, fast, and convenient. The first one requires that the axes α, β, γ of the ellipsoid be determined prior to the calculation of $2V$, while the second one does not. But both of them require the complete extinction curve—or at least a good part of it—which means that many extinctions (or wave-normal : vibration sets) covering a wide angular range, have to be measured and recorded.

In the present work an attempt was made to solve the problem of the determination of $2V$ with the smallest possible number of experimental extinctions; this number turned out to be four.

On the number of experimentally recorded extinctions required for determining the optic axes

Consider a biaxial crystal mounted on a spindle stage (or one-axis stage goniometer), which is set with its rotation axis perpendicular to the axis of the polarizing microscope. We know that its extinction curve will depend on the relative directions of the two optic axes and the spindle-stage axis (Joel and Garaycochea, 1957; Tocher, 1962). Indeed, Garaycochea and Wittke (1963) showed that on a unit sphere the equation of the extinction curve in terms of the three unit vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{r}_0$, respectively parallel to the two optic axes and the spindle-stage axis, is:

$$2(\mathbf{a}_1.\mathbf{r})(\mathbf{a}_2.\mathbf{r})(\mathbf{r}_0.\mathbf{r}) = (\mathbf{a}_1.\mathbf{r}_0)(\mathbf{a}_2.\mathbf{r})+(\mathbf{a}_2.\mathbf{r}_0)(\mathbf{a}_1.\mathbf{r}), \quad (1)$$

where $\mathbf{r}$ is a unit vector from the origin to any point P of the extinction curve.

This equation could be used to determine the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ relative to $\mathbf{r}_0$ and the recorded vibration (extinction) directions $\mathbf{r}_1$, $\mathbf{r}_2$, ..., $\mathbf{r}_n$, . . ., provided that a sufficient number of the latter have been measured and plotted. Since $\mathbf{a}_1$ and $\mathbf{a}_2$ are unit vectors only two components (direction cosines) have to be determined for each of them. The number of unknowns is thus four, which means that four independent extinctions have to be measured and plotted, and as the two vibration directions associated to any given wave-normal are not independent, four different wave-normals are required. The corresponding equations are actually unsolvable in practice unless one uses an electronic computer. But this is not really necessary because a graphical construction can solve the problem, as is shown below. From the purely mathematical point of view the problem has many solutions (a finite number, however). But from the physical point of view the solution is unique: the ellipsoid has only two circular sections. Therefore, if the graphical construction leads to a solution, then this is the solution to the problem.

Special cases arise when P_0 is on one of the three principal planes of the ellipsoid, on one of the circular sections, or coincides with α, β, γ, A_1, or A_2.

Stereographic solution to the problem of locating the two optic axes from four independent measured extinctions

As was mentioned above, Tocher's (1962) procedure will be applied by requiring that for each of the four waves the two optic axes satisfy the relation: $\angle A_1 N_i V_i = \angle A_2 N_i V_i \quad (i = 1, 2, 3, 4)$.

In order to use this condition for determining A_1 and A_2, one could do the following construction (given by Tocher, 1962, p. 55) for each of the four waves:

Consider the great circle $N_1 V_1$ (or $N_1 V'_1$). Pairs of great circles are drawn through N_1 symmetrically disposed to $N_1 V_1$ (or $N_1 V'_1$), for instance at intervals of 10° in this first stage. Any pair of these great circles can be called the pair m, where the index m refers to the angular distance to $N_1 V_1$; so that the pair 20 is the pair of circles that form an angle of 20° with the circle $N_1 V_1$, one circle on each side of it.

A few definitions will be useful for the present discussion. Of any two points that are on any pair m (one on each circle of this pair) we shall say that they belong to the class M. In other words, two points belong to the class M if the two great circles that join them with N_1 form equal angles with $N_1 V_1$.

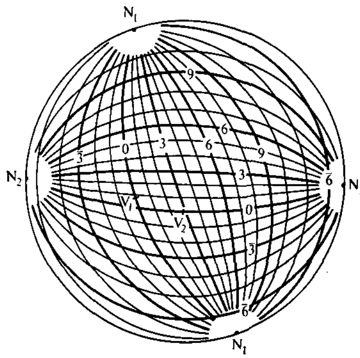
The same construction would be required for the other three waves, with pairs of great circles symmetric relative to $N_2 V_2$, $N_3 V_3$, and $N_4 V_4$. The pairs of circles will be labelled n, p, and q respectively, so that, for instance, a pair of points belong to the class Q if they are on a pair of circles through N_4—one on each—that form equal angles with the circle $N_4 V_4$.

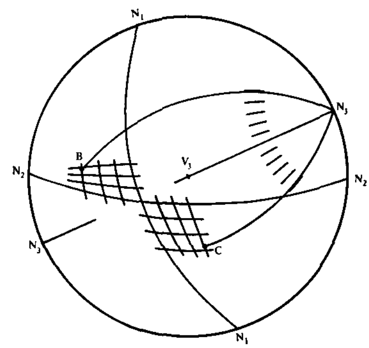
Of any pair of points that belong to both class P and class Q we shall say that they belong to the class PQ (class PQ is thus the intersection of the classes P and Q). Such a pair of points lie on great circles that are symmetrically located relative both to $N_3 V_3$ and to $N_4 V_4$. These definitions can be summarized, and generalized, in the following way. Two points B and C belong to the class M (or N, P, or Q) if they satisfy the condition $\angle BN_i V_i = \angle CN_i V_i$ for $i = 1$ (or $i = 2, 3, 4$); they belong to the class MN if they satisfy this condition for the two wave-normals $i = 1$ and 2; they belong to the class MNP if they satisfy it for the three wave-normals $i = 1$, 2, and 3; and finally to the class $MNPQ$ if they satisfy it for the four wave-normals ($i = 1, 2, 3, 4$).

Consequently, the problem of locating the two points A_1 and A_2 amounts to locating a pair of points that belong to the class $MNPQ$. At first sight it would seem very difficult to find such a pair of points. However, this is not so, and one possible way of finding the solution is as follows:

The grid of circles (fig. 1) through N_1 and N_2, and symmetric relative to $N_1 V_1$ and $N_2 V_2$ respectively, is drawn at intervals of 10°. They can be labelled 0, 1, 2, ..., 8, 9, $\bar{8}$, $\bar{7}$, ..., $\bar{1}$, 0; where 3 and $\bar{3}$ stand for the two circles at 30° to either $N_1 V_1$ or $N_2 V_2$. $N_1 V_1$ and $N_1 V_1'$ are the circles $m = 0$ and $m = 9$, while $N_2 V_2$ and $N_2 V_2'$ are the circles $n = 0$ and $n = 9$. The intersections of any two circles of this grid will have two indices and we shall have points such as 08, $2\bar{3}$, $\bar{4}6$, $5\bar{7}$, etc.; the point $\bar{4}6$ is the intersection of the circle $m = \bar{4}$ $(= -40°)$ with the circle $n = 6$ $(= +60°)$. The points $\bar{4}6$ and $4\bar{6}$ constitute a pair that belongs to the class MN, and it is clear that one can find in the projection an infinity of such pairs of points of class MN when both m and n cover the range 0°...90°.

The next step is to use the third wave-normal N_3, with its vibration direction V_3 (or V'_3). Consider the pair of circles p, that is, the pair of circles through N_3 at angles $+p°$ and $-p°$ from $N_3\,V_3$. Their points, taken in pairs, belong to the class P. If it is possible to find on a certain pair p two points (one on each circle) such that they also belong to the class MN, then those two points will belong to the class MNP, as they

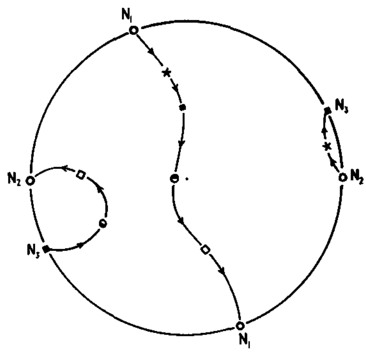


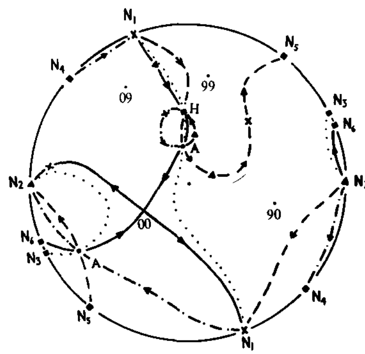


FIGS. 1 and 2: FIG. 1 (left). The grid mn of great circles at 10° intervals, based on two vave-normal: vibration sets $N_1\,V_1$ and $N_2\,V_2$. (In figs. 1–4 the pole of the spindle-stage axis is in the centre of the projection.) FIG. 2 (right). Two points, B and C, that belong to the class MNP. Their indices are $\overline{30}$, 28, 49 and 30, $\overline{28}$, $\overline{49}$ respectively. The grid is the same as that of fig. 1.

will satisfy the condition $\angle BN_i\,V_i = \angle CN_i\,V_i$ for the three wave-normals considered so far. As an example, the points B and C of fig. 2 are on the circles $p = 49$ and $p = -49$, and they are also the points $\overline{30}$, 28 and 30, $\overline{28}$ of the grid mn. It can be seen that at this stage it becomes necessary to think of the circles and the points as having two-figure indices indicating their location in degrees, the indices of B being $m = \overline{30}$, $n = 28$, $p = 49$. A convenient way of finding such a pair of points is as follows: the mn grid, drawn on tracing paper, is rotated until the point N_3 falls on one of the two points of the primitive where the great circles of the stereographic net intersect. In this way, the great circles of the net can be used as pairs p, without drawing them on the tracing paper. Now, on pairs p taken at convenient intervals—say 10° to begin with—pairs of points of class MN are searched for. Or else, on convenient pairs of circles m (or n) points of class NP (or MP) are searched for. Interpolation is usually necessary. In the example mentioned above $m = \overline{30}$ (and 30) was chosen, and by inspection of the projection it was found that the interpolation had to be done between $n = 20$ and 30 ($\overline{20}$ and $\overline{30}$). Finally, $n = 28$ and $p = 49$ were found (fig. 2). This process of interpolation is quite simple, and the search becomes actually easier for the next pair, for instance $n = 20$, m between $\overline{20}$ and $\overline{30}$, where the pair $\overline{26}$, 20, 40 is found.

If this operation is repeated at convenient intervals, two loci are obtained for the points of class MNP, each pair of points having one





FIGS. 3 and 4: FIG. 3 (left). The trisogon $N_1\,N_2\,N_3$, or loci for the pairs of points that belong to the class MNP. The circles, stars, squares, and arrows indicate the correspondence between points of a pair. The two white squares correspond to points B and C of fig. 2. FIG. 4 (right). Four different trisogons: $N_1\,N_2\,N_3$ (dotted line), $N_1\,N_2\,N_4$ (strokes and dots), $N_1\,N_2\,N_5$ (broken line), $N_1\,N_2\,N_6$ (continuous line). The triangles, squares, crosses, and arrows indicate the correspondence between points of a pair. Of the five points of intersection, N_1, N_2, and H do not include a pair; A and A constitute a pair and they are the poles of the optic axes.

point on one locus and the other point on the other locus (fig. 3). It should be noted that a given point of one of these loci cannot be combined arbitrarily with any point of the other; every particular point on one of them is associated with a very definite point of the other. In order to show the correspondence between the two loci in fig. 3, some pairs of points are indicated by means of little squares, circles, stars, and arrows. In fact, each pair of points of class MNP, as they are plotted, should be given some arbitrary number, letter, or symbol.

It will be convenient to give a name to this pair of loci, and the name *trisogon* suggests itself, as the pairs of points on it are determined by three pairs of equal angles relative to $N_1\,V_1$, $N_2\,V_2$, and $N_3\,V_3$. One can thus refer to this trisogon as the trisogon MNP, or the trisogon $N_1\,N_2\,N_3$. In the example shown in fig. 3 the trisogon $N_1\,N_2\,N_3$ has two branches: one going through N_1 for one point of the pair (this is a centrosymmetrical curve that goes round the sphere), and the other branch going through N_2 and N_3 for the other point of the pair (this actually consists of a smaller closed curve and its inverse).

The accuracy of a trisogon will depend on the intervals at which the pairs of points are determined. At this stage no great accuracy is required, and just a sketch of the trisogon will suffice. Refinement can take place later around the optic axes, once they have been approximately found.

In order to find on the trisogon the pair of points of class $MNPQ$, that is, the optic axes, one can proceed as follows: The trisogon $N_1 N_2 N_3$ is superimposed on the circles q. For this purpose the projection is rotated until the point N_4 coincides with the point of the primitive where the great circles of the net intersect; these circles can thus be used as pairs q. Now, find on the trisogon $N_1 N_2 N_3$ a pair of MNP points (either plotted or interpolated) such that they belong to class Q (equal q values). This pair of points will be close to the true positions of the optic axes. In the immediate vicinity of these points the original $N_1 N_2 N_3$ trisogon can then be carefully refined and a small portion of another trisogon $N_1 N_2 N_4$ can be drawn. This $N_1 N_2 N_4$ trisogon is constructed using the original mn grid with the q circles instead of the p ones. The intersections of the two trisogons then give the pair of points of class $MNPQ$, that is, the projections A_1 and A_2 of the two optic axes.

It can be seen immediately how the accuracy of the result can be increased if one has measured and plotted more than four vibration directions. Additional partial trisogons such as $N_1 N_2 N_5$, $N_1 N_2 N_6$, etc. can then be drawn in the critical areas, around the optic axes. The trisogons will intersect each other (ideally in a pair of points) over a limited area in the interior of which the best solution can be determined as that which departs least from the loci. It can be seen that it is convenient to use the two sharpest extinctions for the grid mn.

It is obvious that once the points A_1 and A_2 have been obtained, the determination of the three principal axes of the ellipsoid follows easily, except that it is not possible, without some further information, to decide which of the two bisectrices is α and which is γ (Joel, 1963).

Example and discussion

The stereographic procedure outlined above was tried with experimental extinction measurements and with points taken from a theoretical extinction curve. The latter seems to be of more interest for the present paper, in order to learn something about the accuracy that can be achieved in the determination of the two optic axes from a few good extinction measurements. As to the effect that inaccurate (not sharp) extinctions may have on the final result, it seems difficult to give general rules; in any case, it would be advisable to spend some time over the measurements and to use only the sharpest extinctions.

Fig. 4 shows four typical trisogons that were drawn with data (wave-normals and vibration directions) taken from a theoretical extinction curve ($2V = 90°$). The grid *mn*—not shown in Fig. 4—is the same grid based on N_1 and N_2 shown in fig. 1 and used in figs. 2 and 3. It can be seen that the trisogons shown in fig. 4 have quite different shapes. Three distinct cases can be recognized:

A trisogon with one locus for one of the optic axes A, consisting of a centrosymmetric curve that runs round the sphere, and a separate locus for the other axis A, consisting of two smaller closed curves that are centrosymmetric to each other. This is the case of the trisogon $N_1\,N_2\,N_3$ of fig. 3 (also shown as a dotted line in fig. 4) and the trisogon $N_1\,N_2\,N_4$ drawn in strokes and dots in fig. 4.

Or a trisogon consisting of one continuous (centrosymmetric) curve running round the sphere; part of it is the locus for one axis and the remainder is the locus for the other axis. This is the case of the trisogon $N_1\,N_2\,N_5$ shown in fig. 4 in broken line.

Thirdly, a limiting case between these types can arise, as shown by the trisogon $N_1\,N_2\,N_6$ drawn in fig. 4 in a continuous line. It can be considered as the three curves of the first type having two common points, one in each hemisphere, or the curve of the second type having two intersections, one in each hemisphere.

This limiting case arises when N—the third wave-normal used for drawing the trisogon N_1N_2N—is chosen in such a way that the great circle NV goes through any of the points 00, 09, 90, or 99 of the grid *mn*. In each of these four cases, the intersection on the trisogon is actually the point 00, 09, 90, or 99 respectively. In the example of fig. 4 it is the point 00. These special positions of N can be labelled N_{00}, N_{09}, N_{90}, and N_{99} respectively, so that in fig. 4, N_6 corresponds to N_{00}.

The points N_{00}, N_{09}, N_{90}, and N_{99} divide the projection circle into eight arcs. When a wave-normal is chosen on four of them (alternate ones) a trisogon of the first type arises; and if it is taken on the other four, then a trisogon of the second type is obtained.

In order to show the correspondence between the points of a pair on the trisogons of fig. 4, some typical pairs have been indicated by squares, triangles, crosses, and arrows. It will be seen that all the trisogons intersect at five points, N_1, N_2, H, A, and A. The last two constitute a pair, and they are the optic axes, while the first three points do not include a pair. Indeed, it is obvious that neither N_1 nor N_2 can be an optic axis, for if either of them were, one would not have been able to record an extinction position; however, we note that the corresponding points to N_2 (triangle) on the other trisogons, also marked by triangles, while they do not coincide, all lie on the great circle $N_1 H$, while the corresponding points to N_1 (cross) all lie on $N_2 H$. As to the point H (square), the corresponding points on each of the four trisogons are N_3, N_4, N_5, and N_6, and it follows that H cannot be an optic axis. The problem therefore has no ambiguity, in spite of these three spurious intersections, which arise as a consequence of choice of N_1 and N_2 as the base of the grid mn (fig. 1), so that they are automatically common to all four trisogons, while the third point, H, is also fixed by the choice of N_1 and N_2, as follows: if m and n are the indices of the great circle $N_1 N_2$ (the primitive in figs. 1 to 4) relative to $N_1 V_1$ and $N_2 V_2$ respectively (see fig. 1), then the indices of H are $\overline{m}$ and $\overline{n}$. In the example shown $m = -54°$ and $n = -69°$, so that H has the indices 54, 69.

It thus follows that the two optic axes can in fact be determined by the procedure outlined in this paper as the pair of points at which two trisogons intersect. For the purpose of checking the result and attaining higher accuracy, more than two trisogons can be used.

It is convenient to draw the grid mn in one colour, say black, and to use several colours, letters, numbers, or symbols for marking successive pairs of points on any of the trisogons. The most difficult step would probably seem to be the finding of the first few pairs of points of a trisogon $N_1 N_2 N$. This can be greatly facilitated in the following way.[1] To begin with, N and H constitute one pair of points of the trisogon (the determination of H is explained in the preceding paragraph); and there are two more pairs that are also very easy to determine: N_1 will be paired with a point lying on the great circle $N_2 H$, while N_2 will form a pair with a point on the great circle $N_1 H$. Further pairs can then be searched for in the neighbourhood of any of these three pairs. One soon learns how to avoid exploring regions that are not required. In this respect fig. 4 has a high degree of redundancy.

For the theoretical example shown in fig. 4, with not more than the usual precautions and care, and with a 20 cm diameter net, the angle $2V$ was determined with an error of not more than half a degree. Of course the experimental errors, including those due to possible lack of sharpness of any of the observed extinctions, will add to the error. This effect can be minimized by using only the sharpest extinction readings.

The present method for determining the two optic axes—and the three principal axes of the ellipsoid—of a biaxial crystal is not being advocated as a handy one, as the stereographic constructions are indeed

[1] I am grateful to Dr. F. E. Tocher for suggesting this approach via the point H.

lengthy and slow, and can be quite tiring. But it may be useful for cases in which only interrupted ranges of the extinction curve are available. Furthermore, it is felt that it is of some theoretical interest as it shows how the directions of the two optic axes (in other words, the optic axial angle $2V$ and the orientation of the ellipsoid) are physically determined by any four independent extinction measurements. Finally, it can also be used in connexion with the universal stage, which could be useful for selecting a convenient rotation axis in the crystal, that is, a convenient plane for the wave-normals. There is no real need for these wave-normals to be all in the same plane; but if they are non-coplanar, then the stereographic constructions become more difficult.

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Appendix. A direct mathematical formulation

The following mathematical formulation follows directly from the Biot–Fresnel theorem; it is more general than the treatment given in the foregoing sections, as it does not require the wave-normals to be coplanar.

The equation of the plane that contains the optic axis A_1 and the wave-normal N is:

$$\mathbf{a}_1\times\mathbf{n}.\mathbf{r} = 0 \tag{2}$$

and that of the plane through A_2 and N:

$$\mathbf{a}_2\times\mathbf{n}.\mathbf{r} = 0. \tag{3}$$

The equation of the two planes through N that bisect the dihedral angle formed by these planes is:

$$\frac{\mathbf{a}_1\times\mathbf{n}.\mathbf{r}}{|\,\mathbf{a}_1\times\mathbf{n}\,|} \pm \frac{\mathbf{a}_2\times\mathbf{n}.\mathbf{r}}{|\,\mathbf{a}_2\times\mathbf{n}\,|} = 0. \tag{4}$$

This equation is satisfied by any wave-normal $\mathbf{n}_i$ and its corresponding vibration directions $\mathbf{r}_i$ or $\mathbf{r}'_i$, and it can in fact be used (at least in theory) for determining the optic axes $\mathbf{a}_1$ and $\mathbf{a}_2$ from observed extinction measurements. As $\mathbf{a}_1$ and $\mathbf{a}_2$ are unit vectors, only two components have to be determined for each; in all, four unknowns. Therefore, one uses four wave-normals $\mathbf{n}_i$ ($i = 1, 2, 3, 4$) with one of the two vibration directions $\mathbf{r}_i$ associated with each of them, and sets up four equations of the form:

$$|\mathbf{a}_2\times\mathbf{n}_i|\mathbf{a}_1\times\mathbf{n}_i.\mathbf{r}_i = \pm|\mathbf{a}_1\times\mathbf{n}_i|\mathbf{a}_2\times\mathbf{n}_i.\mathbf{r}_i. \tag{5}$$

The triple products $\mathbf{a}\times\mathbf{n}_i.\mathbf{r}_i$ can be rearranged as $\mathbf{n}_i\times\mathbf{r}_i.\mathbf{a}$; and equations (5) become:

$$|\mathbf{a}_2\times\mathbf{n}_i|\mathbf{n}_i\times\mathbf{r}_i.\mathbf{a}_1 = \pm|\mathbf{a}_1\times\mathbf{n}_i|\mathbf{n}_i\times\mathbf{r}_i.\mathbf{a}_2. \tag{6}$$

But the product $\mathbf{n}_i\times\mathbf{r}_i$ is a unit vector perpendicular to $\mathbf{n}_i$ and $\mathbf{r}_i$ and has therefore the direction of the second vibration associated with $\mathbf{n}_i$. Therefore, $\mathbf{n}_i\times\mathbf{r}_i$ can be replaced by $\mathbf{r}_i$; and the equations to be solved are, together with $\mathbf{a}_1^2 = \mathbf{a}_2^2 = 1$:

$$(\mathbf{a}_2\times\mathbf{n}_i)^2(\mathbf{a}_1.\mathbf{r}_i)^2 = (\mathbf{a}_1\times\mathbf{n}_i)^2(\mathbf{a}_2.\mathbf{r}_i)^2 \quad (i = 1, 2, 3, 4). \tag{7}$$

The unknown components (a_{11}, a_{12}, and a_{13} of $\mathbf{a}_1$; a_{21}, a_{22}, and a_{23} of $\mathbf{a}_2$) appear in products such as $a_{11}^2\,a_{21}^2$, $a_{12}\,a_{13}\,a_{22}^2$, $a_{12}\,a_{13}\,a_{21}\,a_{22}$, etc., so that it is not a practical proposition to solve these equations, but the problem can be tackled graphically and one way of obtaining the answer is the stereographic solution given above.

Equations (7) can also be deduced from a diagram, as follows:

In the stereographic projection, fig. 5, the arcs $M_1 V$ and $M_2 V$ are equal, and therefore: $\cos M_1 V = \cos M_2 V$. But in the right-angled triangles $A_1 M_1 V$ and $A_2 M_2 V$ $\cos A_1 V = \cos A_1 M_1 \cos M_1 V$ and $\cos A_2 V = \cos A_2 M_2 \cos M_2 V$; therefore:

$$\cos A_1 V/\cos A_2 V = \cos A_1 M_1/\cos A_2 M_2 = \sin NA_1/\sin NA_2$$

since $NA_1+A_1 M_1 = NA_2+A_2 M_2 = 90°$. This can be written (the + sign applies to the vibration V shown in the figure, the − sign to the other vibration):

$$\frac{\mathbf{a}_1.\mathbf{r}}{\mathbf{a}_2.\mathbf{r}} = \pm\frac{|\mathbf{a}_1\times\mathbf{n}|}{|\mathbf{a}_2\times\mathbf{n}|}, \qquad (8)$$

from which equations (7) follow immediately.

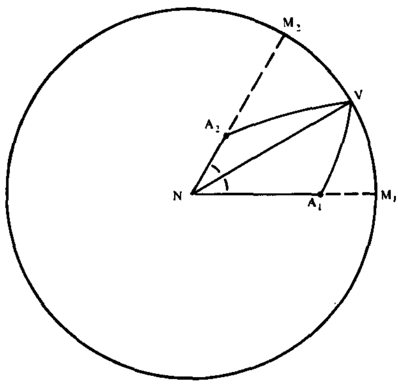

FIG. 5. Projection (with the wave-normal N in the centre) to illustrate the relation between the angles NA_1, NA_2, $A_1 V$, $A_2 V$.

If the wave-normals $\mathbf{n}$ are restricted to be coplanar, all of them being perpendicular to the vector $\mathbf{r}_0$ (spindle-stage axis), then in equations (7) and (8) $\mathbf{n}$ can be replaced by $\mathbf{r}_0\times\mathbf{r}$ (because $\mathbf{n}$ is in this case parallel to $\mathbf{r}_0\times\mathbf{r}$); and after some manipulation one arrives at equation (1) (p. 771). It is interesting to notice that in this way the equation of the extinction curve has been derived directly from the Biot–Fresnel theorem.

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