Direct stereographic determination of the optic axes from a few extinction measurements: a progressive elimination technique

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Summary. A new method of determining the optic axes from a few extinction measurements is presented. It is based on the progressive elimination of those parts of the stereogram in which it is obvious that the optic axes cannot lie. It has two applications: as a valid method in its own right, if applied systematically; and as a reconnaissance technique. The second application permits, for example, the recently presented trisogon technique to be rapidly and economically applied over a very small fraction of the stereogram instead of over its whole area.

DETERMINATION of the optic axes of a biaxial crystal by most established extinction techniques (Wilcox, 1960; Tocher, 1962; Joel, 1963; Garaycochea and Wittke, 1964) is normally possible only if the extinction data delineate an extinction or other curve sufficiently complete for identification of either the indicatrix axes, α, β, γ, or some other critical points or planes. The method recently proposed by Joel (1964), using a minimum of only four extinction measurements, is of great interest in that it renders usable extinction data that hitherto have had to be discarded for lack of completeness. Joel, however (*ibid.*, p. 777), appears to have considered his trisogon technique likely to be of academic interest only or to become, at best, a last-resort method of optic axis determination. Investigation of several special cases of trisogons has, however, brought to light the method advocated in the present paper—a method calculated to bring the trisogon approach, if not within the realm of the everyday technique, at least within the optical repertoire of many who might otherwise be repelled by the apparent complexity and length of the construction. The method reduces drastically the area of the stereogram over which the initial trisogon need be constructed and is based upon the same few extinction measurements as the trisogons themselves. In addition, the approach is radically different from any hitherto used in this field: it is aimed not so much at locating the optic axes as at the progressive elimination of those parts of the stereogram in which it is obvious that the optic axes cannot lie.

Although this method was originally evolved with 'pre-trisogon reconnaissance' in mind, it is shown in a subsequent section that it has potentialities in its own right as a method for determination of the optic axes from a relatively small number of extinction measurements.

The basis of the method

For a crystal fragment mounted on a spindle stage or a one-circle stage goniometer, extinction measurements are, in general, obtainable for a series of wave-normals, N_1, N_2, N_3,..., N_n, at right angles to the spindle axis, P_0. In the case of any general wave-normal of this type, say N_1, the relationships relevant to the technique outlined in the present paper are depicted in fig. 1. The vibrations, V_1 and V_1', on the polar ($P_0\,V_1$) and equatorial ($U V_1'\, U$) extinction curves (Joel and Garaycochea, 1957; Tocher, 1962) respectively, define, together with N_1, the planes $N_1\,V_1$ and $N_1\,V_1'$; and these planes, by the Biot-Fresnel construction, bisect the acute and obtuse angles between the planes, $N_1\,A_1$ and $N_1\,A_2$, through N_1 and the two optic axes, A_1 and A_2. However, of the two planes, $N_1\,V_1$ and $N_1\,V_1'$, only one, $N_1\,V_1$, will fall between $N_1\,A_1$ and $N_1\,A_2$ on the upper hemisphere provided, of course, that neither A_1 nor A_2 lies on the primitive circle. On this basis, it is clear that one optic axis (A_1 in fig. 1) must fall within the acute angle between the plane $N_1\,V_1$ and the primitive circle. The other axis (A_2 in fig. 1) must therefore fall within an area of the same angular width on the other side of plane $N_1\,V_1$, i.e. between planes $N_1\,V_1$ and H_1,[1] where the latter and the primitive circle[2] are equally inclined to plane $N_1\,V_1$.

Plane H_1 clearly divides the stereogram into two parts: that containing plane $N_1\,V_1$ and both optic axes, and, secondly, the remainder of the stereogram (ruled in fig. 1). The latter may be eliminated in the search for optic axes.

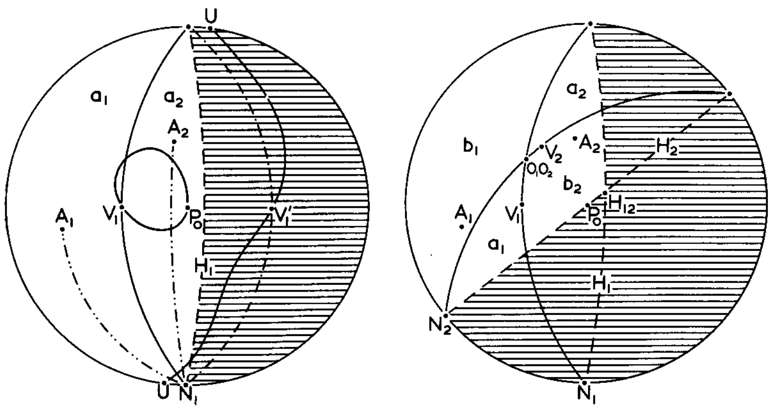
Application

Reconnaissance. If only four extinction measurements, the minimum necessary for the determination of the optic axes (Joel, 1964), are available then the above considerations may be profitably used as a means of reducing to a minimum the area over which a search need be made for

[1] Plane H_1 is referred to in this manner since it contains the point H, as defined by Joel (1964, p. 777). However, H is of variable position within the plane and is definable only in terms of two wave-normals. See figs. 2 *et seq.*

[2] The primitive circle can likewise be referred to as plane H_0: it is the type H plane associated with N_0, the wave-normal whose polar curve vibration, V_0, coincides with P_0.

the optic axes. If the four wave-normals, N_1, N_2, N_3, N_4, and their associated pairs of vibrations, V_1 and V_1', V_2 and V_2', etc., are plotted on a stereogram with P_0 central there can, in general, given sufficient familiarity with the shape of extinction curves (Joel and Garaycochea, 1957; Wilcox, 1960; Tocher, 1962; Fisher, 1962; Joel, 1963; Garaycochea and Wittke, 1964), be little doubt regarding which member of

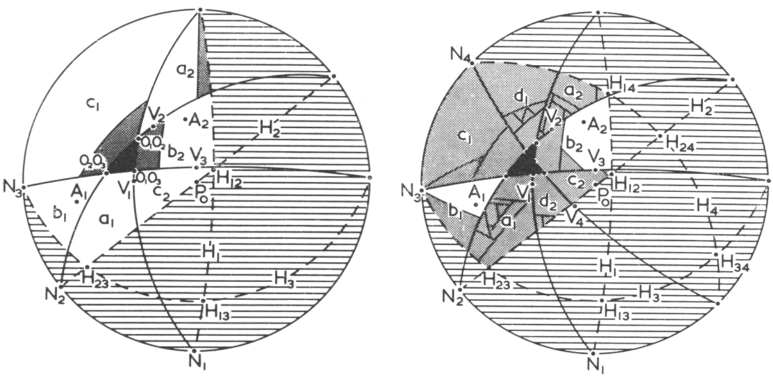


FIGS. 1 and 2: FIG. 1 (left). General stereogram with the axis of rotation, P_0, of the crystal as centre, showing: P_0 V_1, the polar, and $U V_1' U$, the equatorial extinction curves containing V_1 and V_1' respectively, the vibration directions associated with wave-normal N_1; planes N_1 V_1 and N_1 V_1' bisecting, internally and externally respectively, the angle A_1 N_1 A_2 subtended by the optic axes, A_1 and A_2, at N_1; plane H_1 making, at N_1, the same acute angle with plane N_1 V_1 as does the primitive circle; the initial elimination area (ruled) to the right of plane H_1; the initial search area to the left of plane H_1; and sub-areas a_1 and a_2 of the initial search area, each containing one optic axis. FIG. 2 (right). Stereogram, centre P_0, showing: two wave-normals, N_1 and N_2, with associated polar curve vibrations, V_1 and V_2; planes H_1 and H_2 defining the point H_{12}; the initial elimination area (ruled) and the initial search area (blank); planes N_1 V_1 and N_2 V_2 intersecting in the point 0_1 0_2 and defining the sub-area pairs, a_1, a_2, and b_1, b_2, of the initial search area; and the optic axes, A_1 and A_2, each in a different member of the same sub-area pair.

each vibration pair lies on the polar curve. The vibrations on the polar curve are named V_1, V_2, V_3, V_4, the subscript in each case being the same as that of the associated wave-normal. The vibrations on the equatorial curve, V_1', V_2', V_3', V_4', may be ignored.

When the N_1 V_1 and H_1 planes are drawn on the stereogram (fig. 1) each serves a specific function: H_1 defines the search area—that part of the stereogram within which the optic axes lie, on the V_1 side of H_1; and N_1 V_1 divides the search area into two sub-areas, a_1, containing one optic axis, and a_2, containing the other.

Addition of the $N_2\,V_2$ and H_2 planes (fig. 2) immediately reduces the search area by excluding all of that part of the stereogram on the elimination side of each of the planes, H_1 and H_2. Moreover, the reduced search area is now divided by the planes, $N_1\,V_1$ and $N_2\,V_2$, into four sub-areas, only two of which will, in general, contain the optic axes. Since the optic axes must lie one on each side of plane $N_1\,V_1$ and also



FIGS. 3 and 4: FIG. 3 (left). Stereogram, centre P_0, showing: three wave-normals, N_1, N_2, N_3, and associated polar curve vibrations, V_1, V_2, V_3; planes H_1, H_2, H_3 intersecting in points H_{12}, H_{23}, H_{13}; planes $N_1\,V_1$, $N_2\,V_2$, $N_3\,V_3$ intersecting in points $0_1\,0_2$, $0_2\,0_3$, $0_1\,0_3$ and defining the 00 elimination area (black); increased initial elimination area (ruled); division of the initial search area into three sub-area pairs, a_1, a_2, b_1, b_2, and c_1, c_2; the beginnings of progressive elimination (stippled) in sub-areas c_1, a_2, and b_2 with respect to wave-normals N_1, N_2; and the optic axes, A_1, A_2. FIG. 4 (right). Stereogram, centre P_0, showing: four wave-normals, $N_1 ... N_4$, and associated polar curve vibrations, $V_1 ... V_4$; planes $H_1 ... H_4$ intersecting in points $H_{12} ... H_{34}$; planes $N_1\,V_1 ... N_4\,V_4$ intersecting in points $0_1\,0_2 ... 0_3\,0_4$ (not indexed) and defining the 00 elimination area (black); increased size of both the initial elimination (ruled) and 00 elimination areas; division of the reduced initial search area into four sub-area pairs, a_1, a_2, b_1, b_2, c_1, c_2, and d_1, d_2; and around the optic axes, A_1, A_2, the residual search area (blank) derived by progressive elimination of the stippled portion of the initial search area (successive stages indicated by lines).

one on each side of plane $N_2\,V_2$, the four sub-areas can be immediately arranged in two pairs: an optic axis lies in each of sub-areas a_1 and a_2, or in each of b_1 and b_2.

It may be noted here that the intersection of planes H_1 and H_2 defines the point H_{12}. This point corresponds to the point H in fig. 4 of Joel's 1964 paper and is a point on every trisogon based on the two wave-normals, N_1 and N_2, and the associated great circle grid, i.e. on every trisogon, $N_1\,N_2\,N$. Points H_{13}, H_{23}, H_{14}, etc., appearing in subsequent figures have a similar connotation. In the same way, intersection of planes N_1 V_1 and N_2 V_2 defines the point 0_1 0_2, corresponding to Joel's point 00 for wave-normals N_1 and N_2. The subscripts are added here since, in general, no two pairs of wave-normals have the same point 00. Points 0_1 0_3, 0_2 0_3, 0_1 0_4, etc., in subsequent figures are indexed on the same basis.

Plotting of the N_3 V_3 and H_3 planes (fig. 3) further reduces the initial search area. This is due not only to exclusion of that part of the search area of fig. 2 now on the elimination side of plane H_3 but also to exclusion of the triangle, $0_1 0_2$ $0_2 0_3$ $0_1 0_3$ (black in fig. 3), within the initial search area, in which planes N_1 V_1, N_2 V_2, and N_3 V_3 intersect—this triangle, the 00 elimination area, cannot be paired with any other part of the initial search area. The initial search area is now subdivided into three pairs of sub-areas, a_1 and a_2, b_1 and b_2, c_1 and c_2. The members of each pair are diametrically opposed across the 00 elimination area and are bounded laterally by the same pair of NV planes. In this connexion the use of a different colour for each set of NV and H planes is found to be of considerable assistance: the former are drawn as continuous lines, the latter as broken lines.

At this point it is possible, merely by inspection and without any further extinction data, to reduce the search area still further. With plane N_1 V_1 as zero meridian, the maximum and minimum angles, as measured at N_1, subtended by each of the areas a_1 and a_2 are measured. Since the optic axes must, at N_1, subtend equal angles with plane N_1 V_1 (fig. 1), they can only fall within those parts of sub-areas a_1 and a_2 that do subtend equal angles with plane N_1 V_1; the above maximum and minimum angles are therefore equalized by eliminating those parts of either sub-area (stippled in fig. 3) which fall outside the angular range common to both. Sub-areas b_1, b_2, and c_1, c_2 are treated in a similar fashion. The process is then repeated for wave-normals N_2 and N_3 in the order in which they appear on the primitive circle. Since the elimination of part of any sub-area with respect to any one wave-normal will, in general, affect its angular width with respect to the other wave-normals, the entire sequence is repeated with progressive elimination of more and more of each sub-area. Eventually, after a small number of complete repetitions of the sequence (in some cases only one repetition may be necessary), progressively diminishing eliminations suggest that further repetition would be more time-wasting than profitable.

Addition of the fourth wave-normal, N_4, and its associated planes, N_4 V_4 and H_4 (fig. 4), reduces the search area still further. This reduction is due partly to encroachment on the initial search area of fig. 3 by the plane H_4, partly to an increase in the size of the 00 elimination area, and partly to greatly increased possibilities for progressive elimination within the paired sub-areas, now four in number. This third process results in one or two or, in favourable cases, three pairs of sub-areas being completely eliminated. The residual search area now consists of two continuous units, each embracing small portions of not more than three adjacent sub-areas. In practice, of course, all four NV and H planes are plotted on the stereogram before any progressive elimination takes place —the process can then be completed fairly rapidly.

If no further extinction information is available the optic axes may be accurately located as the points of intersection of at least two of the trisogons, $N_1 N_2 N_3$, $N_1 N_2 N_4$, $N_1 N_3 N_4$, $N_2 N_3 N_4$ (Joel, 1964). Of these, only the first constructed need cover the entire residual search area. The remainder need only be constructed, as Joel suggests, in the immediate vicinity of their mutual intersection points. If, however, a few more extinction measurements are available, it may be found profitable to incorporate the wave-normals and associated NV and H planes in the reconnaissance stereogram in order to reduce still further the residual search area.

If at any time one or more of the NV' planes is used in error to define the associated H planes and elimination areas, the problem becomes, of course, insoluble—the area containing both optic axes is wholly eliminated. However, the detection of such an error is very simple on the basis of the following properties of the initial search area: these are only in evidence if the NV planes have been used. It is a continuous area, divided into twice as many sub-areas as there are NV planes. Near the centre, when more than two NV planes are plotted, is the 00 elimination area—this may, in some cases, be reduced to almost a point. The perimeter is defined by H planes and by part of the primitive circle: of these, the latter forms the outer boundary to only one sub-area, except when one of the wave-normals is N_0—the sub-areas immediately on either side of N_0 then abut on the primitive circle.

In conclusion, this method may equally well be employed for reconnaissance purposes in association with other accurate methods of optic axial determination, such as those advocated by Wilcox (1960) and Tocher (1962): the search area within the known optic axial plane can be drastically reduced.

Determination of the optic axes. If more than the minimum of four extinction measurements is available and if the operator can choose the wave-normals at will, then the progressive elimination method is capable of employment in determining, within quite narrow limits in favourable cases, the *positions of the optic axes*. As has been suggested, the addition of further wave-normals and associated NV and H planes to the stereogram of fig. 4 serves to reduce the residual search areas around the optic axes until, eventually, given the correct conditions, they are reduced to two points, the optic axes. Experiment has shown that this reduction to two points is, in general, only possible when one of the NV planes contains both optic axes, i.e. when a wave-normal, N, lies in the optic axial plane. On this basis, the problem superficially reduces to that of finding the optic axial plane. In any given case, the number of extinction measurements necessary to achieve this can be kept to a minimum by the use of a simple ranging technique.

As a preliminary, a reconnaissance is made on the basis of four wave-normals which, for maximum efficiency, should be arranged at 45° intervals around the primitive circle, i.e. around the spindle axis, P_0. Progressive elimination will then provide a residual search area embracing, in general, parts of up to three adjacent sub-area pairs—at least one sub-area pair will be completely eliminated. If the residual search area is confined to one sub-area defined by, say, the $N_2\,V_2$ and $N_3\,V_3$ planes (fig. 5), then the optic axial plane must pass between these two planes. A wave-normal, N_5, is selected, approximately bisecting the angle $N_2\,N_3$, and the associated extinction measurement is used to plot plane $N_5\,V_5$—plane H_5 is, at this stage, unnecessary. Progressive elimination will now, in general, eliminate one of the newly defined sub-areas on either side of plane $N_5\,V_5$ and considerably reduce the other. This reduced search area is now approximately bisected by plane $N_6\,V_6$ derived from a suitably chosen wave-normal, N_6, and the procedure repeated as often as is necessary to reduce the search area to the desired dimensions, say 1° or 2° diameter. In general, the desired reduction can be achieved with not more than seven or, at most, eight extinction measurements provided that at least one of the wave-normals is within 2° of the optic axial plane. In the more sensitive cases (see below) the requirements are much less rigorous.

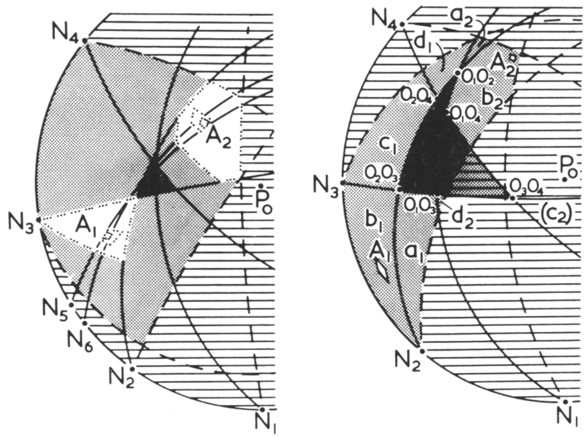
If the residual search area derived from the first four extinction measurements embraces parts of three adjacent sub-areas, the optic axial plane will, in general, pass through the middle one of the three: the procedure is essentially as outlined above. If parts of only two adjacent sub-area pairs constitute the residual search area, then the two parts will normally be unequal. In this case, the optic axial plane will pass, normally, through the sub-area contributing the greater part of the residual search area; but in some of the less sensitive cases (see below) the reverse may be the case. However, that the wrong part of the residual search area has been bisected is soon obvious from the results: one of the new sub-area pairs, the farther from the optic axial plane, is usually completely eliminated; the other is greatly reduced or eliminated; and the unbisected portion of the former residual search area remains virtually undiminished in area. Thus, even with such an erroneous choice of wave-normal, the desired result is partly achieved: the residual search area is much reduced.

Finally, this method of optic axial determination is clearly adaptable for use with the universal stage in those cases where not more than one optic axis can be brought into coincidence with the microscope axis.

Sensitivity

For the minimum of four wave-normals, the sensitivity of this method of optic axial determination is, in general, directly related to the size of the 00 elimination area: the larger this is in relation to the size of the initial search area, the smaller in proportion will be the residual search area. The size of the 00 elimination area is directly related to three factors: the angle $2V$; the proximity of each optic axis to the primitive circle (H_0); and the range over which the four wave-normals, N, are spread on the primitive circle. For general cases, maximum sensitivity occurs when $2V$ is 90°, when P_0 is close to β, and when the four wave-normals, N, are spaced at 45° intervals around the primitive circle. Minimum sensitivity occurs when $2V$ is nearly zero, when P_0 is close to the acute bisectrix, and when the four wave-normals, N, are closely grouped on the primitive circle. The second condition for maximum sensitivity is, of course, that for maximum sensitivity of $2V$ determination by established extinction-curve methods—P_0 near a circular section (Wilcox, 1961; Tocher, 1962, p. 58).

In the more sensitive cases it is particularly important, especially when no colours are employed in the delineation of the NV planes, that the full extent of the 00 elimination area be determined. This involves correct identification and indexing of all six of the 00 points associated with the four initial wave-normals, and is necessary because, in such cases, the 00 elimination area may extend beyond the limits of the initial search area (fig. 6). In such a case, at least one sub-area pair is represented within the initial search area by only one member (c_1 in fig. 6): the other member of the pair (c_2 in fig. 6) only exists in 'negative' form outside the initial search area and, consequently, the first (c_1 in fig. 6) may be immediately eliminated. In such a case, the absolute minimum of four wave-normals (Joel, 1964, p. 770), regardless of whether or not one is close to the optic axial plane, is sufficient to locate each optic axis within a residual area not more than 2° in diameter (fig. 6).



FIGS. 5 and 6: FIG. 5 (left). Partial stereogram, centre P_0, showing: residual search area (within three-dot line) after progressive elimination (stippled) on the basis of four wave-normals, $N_1 ... N_4$, at approximately 45° intervals around the primitive circle; decreased residual search area (within two-dot line) after addition of wave-normal N_5, where N_5 approximately bisects angle $N_2\ N_3$; final residual search area (within dotted line) after addition of wave-normal N_6 such that plane $N_6\ V_6$ approximately bisects the two-dot residual search area. On measurement, $57° < 2V < 63°$. The planes and various intersections are not indexed but conventions established in figs. 1 to 4 have been followed. FIG. 6 (right). Partial stereogram, centre P_0, showing relations in a case of high sensitivity: with four wave-normals, $N_1 ... N_4$, at approximately 45° intervals around the primitive circle, the 00 elimination area is partly (black) within the initial search area and partly (stippled and ruled) within the initial elimination area; sub-area c_2 exists in 'negative' form only, within the initial elimination area; and the residual search areas (blank), produced by progressive elimination (stippled) within the initial search area, give, by measurement, $87{\cdot}5° < 2V < 92{\cdot}5°$ ($2V$ is actually 90°).

In a case of average sensitivity, on the other hand, at least one wave-normal must fall within about 2° of the optic axial plane (fig. 5) before the residual search area will converge sufficiently closely on the optic axes.

Special cases

P_0 coincides with a bisectrix. Here the polar curve degenerates to the point P_0, so that all $N_n\ V_n$ planes are of the type $N_0\ V_0$ and pass through P_0; the 00 elimination area has zero area; and every H_n plane coincides

with H_0, the primitive circle. Consequently, no initial or progressive elimination is possible and the case is insoluble by this method.

P_0 *coincides with* β. Here, both optic axes lie on the primitive circle, so that the undifferentiable $N_n\, V_n$ and $N_n\, V'_n$ planes and all H_n planes coincide with each other and with the primitive circle. The initial search area has zero area (it is linear—the primitive circle) and the case is insoluble by this method.

P_0 *lies on one circular section.* Here, one optic axis, say A_1, lies on the primitive circle so that every $N_n\, A_1$ plane coincides with the primitive circle and, by the basis of the method (fig. 1), every $N_n\, A_2$ plane coincides with the corresponding H_n plane. Every H_n plane therefore passes through optic axis A_2 and all H points coincide with A_2. This case is therefore immediately soluble by inspection for optic axis A_2. Optic axis A_1, however, can have any position on that part of the primitive circle that contributes to the perimeter of the initial search area. Although this part of the primitive circle can be reduced by the ranging technique, progressive elimination cannot derive a unique solution for A_1.

P_0 *has any position on a principal plane of the indicatrix other than as above.* This has all the properties of a general case and cannot be regarded as special as far as this method is concerned.

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