FLETCHER'S INDICATRIX AND THE ELECTROMAGNETIC THEORY OF LIGHT

T. C. PHEMISTER, *University of Aberdeen, Aberdeen, Scotland.*

ABSTRACT

The optical scalars and vectors for an inactive, crystalline dielectric are first deduced from the Maxwell equations for the electromagnetic field and it is then shown how these can, in fact, be derived from the Fletcher indicatrix. Attention is drawn to the importance of the focal lines in the geometry of the indicatrix.

In most texts on crystal optics in use amongst mineralogists and crystallographers at the present time the optical properties of crystals are derived from a consideration of the Fletcher indicatrix.[1] This has proved a most useful surface of reference as the primary concern of the mineralogist is with wave-normals, refractive indices and directions of vibration, all of which are readily derived from this simple figure. The enquiring student however always wishes to know how the surface itself is obtained and how it is related to the electromagnetic theory of light, and it is difficult to refer him to any text where the subject is concisely treated. Fletcher himself presents the indicatrix in a purely geometrical form and does not link it up with any specific view regarding the nature of light. In the systematic German texts,[2] although the connection of the surface with the results of the elctromagnetic theory is pointed out, it is a little difficult for the student to disentangle the proof from the other, possibly more fundamental, aspects of optical theory. It seems worth while therefore to present a short statement showing how this reference surface is bound up with the classical theory of the electromagnetic nature of light.

The treatment adopted is, first, to derive in the usual way the optical scalars and vectors using Maxwell's equations, and then to show that these can in fact be derived from the indicatrix. The presentation is much simplified by using the abbreviated notation of Cartesian tensors[3] and, since the simpler theory of the propagation of light in isotropic media is adequately treated in standard works, a knowledge of it is assumed in what follows.

[1] Fletcher L.: The optical indicatrix and the transmission of light in crystals, London (1892).

[2] Pockels, F.: Lehrbuch der Kristalloptik, B. G. Teubner, Leipzig and Berlin (1906).
Szivessy, G.: Handbuch d. Physik, vol. **20**, J. Springer, Berlin (1928).
Born, M.: Optik, J. Springer, Berlin (1933).

[3] Jeffreys, Vide, H.: Cartesian Tensors, Cambridge University Press (1931).

The various quantities are represented by the following symbols:

space co-ordinates	x_i, ξ_i	directions-cosines	l_i, m_i
outward normal	n_i	volume	τ
surface	S	total energy density	w
volume density of charge, e.s.u.	ρ	magnetic energy density	t
electrical energy density	u	magnetic intensity—m.s.u.	H_i
electrical intensity—e.s.u.	E_i	conduction current density	i_i
dielectric displacement	D_i	Poynting energy vector	N_i
magnetic induction—m.s.u.	B_i	dielectric constant	K
direction-cosines of N_i	p_i	velocity of electromagnetic radiation in vacuo	c
wave-length	λ	frequency	ν
period	T	refractive index	n
ray index	r	wave velocity	α
ray velocity	β	principal wave (and ray) velocities	$v'v''v'''$
principal refractive indices	$\begin{cases} n'n''n''', \\ n' < n''' \end{cases}$		
time	t		

Differentiation with respect to time is indicated by a dot over the quantity concerned.

1. The Relations Between the Electromagnetic Quantities at a Point in a Varying Field are given by the Four Maxwell Equations

$$\operatorname{curl} H_i = \frac{4\pi}{c} i_1 + \frac{\dot{D}_i}{c} \qquad \operatorname{div} D_i = 4\pi\rho$$

$$\operatorname{curl} E_i = -\frac{\dot{B}_i}{c} \qquad \operatorname{div} B_i = 0$$

In the classical continuum theory the specific properties with which we are concerned in the transmission of light by non-conducting crystals are the magnetic permeability and the dielectric constant. The first of these we take as unity on the assumption that the crystals are non-magnetic. It remains therefore to investigate the nature of the dielectric constant.

2. Relation Between D_i and E_i

In an isotropic substance D_i coincides with E_i but in an anisotropic material this is no longer the case.

Thus E_1 itself will produce a displacement with components along $0x_1$, $0x_2$ and $0x_3$. Similarly for E_2 and E_3.

Each of the components of the displacement therefore will be the sum of the contributions from each of the components of the intensity, i.e. e.g.,

$$D_1 = K_{11}E_1 + K_{12}E_2 + K_{13}E_3$$

where $K_{12}E_2$ is the contribution of E_2 to the displacement along $0x_1$, and so on.

In this case therefore the dielectric factor is a second order tensor and the displacement is the product of this tensor and E_i thus,

$$D_i = K_{ij}E_j.$$

3. The Energy Relations in an Anisotropic Dielectric

The magnetic and electrical energy densities are given by,

$$t = \frac{H_i^2}{8\pi} \qquad u = \frac{E_iD_i}{8\pi} = \frac{K_{ij}E_iE_j}{8\pi}.$$

Since div $[E,H]_m = H_i \cdot \text{curl } E_i - E_i \cdot \text{curl } H_i$, we have, from the general equations of the electromagnetic field,

$$-\frac{(E_i\dot{D}_i + H_i\dot{B}_i)}{4\pi} = i_iE_i + \frac{c \text{ div } [E, H]_m}{4\pi}$$

which here becomes,

$$K_{ij}E_i\dot{E}_j - \tfrac{1}{2}\dot{H}_i^2 = -c \text{ div } [E, H]_m.$$

Integrating throughout any volume,

$$\frac{1}{4\pi}\int (K_{ij}E_i\dot{E}_j + \tfrac{1}{2}\dot{H}_i^2)d\tau = -\frac{c}{4\pi}\int \text{div } [E, H]_m d\tau = -\frac{c}{4\pi}\int [E, H]_n dS.$$

Taking the right-hand side as the total energy flow across the surface, $(K_{ij}E_i\dot{E}_j)/4\pi$ is the time rate of change of the electrical energy density $(\frac{1}{2}\dot{H}_i^2)/4\pi$ is the time rate of change of the magnetic energy density.

Differentiating with respect to time the expression for u,

$$\frac{K_{ij}(E_i\dot{E}_j + E_j\dot{E}_i)}{8\pi} = \dot{u} = \frac{K_{ij}E_i\dot{E}_j}{4\pi}$$

so that

$$K_{ij}(E_i\dot{E}_j - E_j\dot{E}_i) = 0$$

which can be written in the form,

$$K_{ij}E_i\dot{E}_j - K_{ij}E_j\dot{E}_i = 0.$$

The "dummy" suffixes can be transposed in the second term giving,

$$(K_{ij} - K_{ji})E_i\dot{E}_j = 0.$$

Since this expression must hold for any value of the field strength,

$$K_{ij} = K_{ji}$$

i.e., the dielectric tensor K_{ij} is symmetrical and has only six different components.

4. Transformation of K_{ij} into Normal Form

The equation $K_{ij}x_ix_j = $ (a constant) represents a central conicoid and, since the discriminating cubic equation,

$$|(K_{ij} - \delta_{ij}\lambda)| = 0$$

(where δ_{ij} is the substitution tensor) has in general three distinct roots for λ, there are three mutually perpendicular principal directions. These are given by $l_j\lambda = l_i K_{ij}$. If the x_i are measures of the electric field strengths E_i, then $K_{ij}E_jE_j$ is $8\pi u$ and, since the energy is positive for all systems of E_i, the equation has always a definite form and so must represent an ellipsoid.

Referring the equation to the three mutually perpendicular principal directions $0\xi_i$ as axes

$$K_{ij}x_ix_j = K'\xi_1^2 + K''\xi_2^2 + K'''\xi_3^2 = \text{(a constant)}.$$

K', K'', and K''' are called the principal dielectric constants.

In this normal form,

$$D_1 = K'E_1, \text{ etc.}$$

$$8\pi u = K'E_1^2 + K''E_2^2 + K'''E_3^2 = \frac{D_1^2}{K'} + \frac{D_2^2}{K''} + \frac{D_3^2}{K'''}.$$

5. Electromagnetic Waves in a Crystalline Dielectric

The differential equations for the E_i are,

$$\frac{K}{c^2}\frac{\partial^2 E_i}{\partial t^2} = \Delta^2 E_i - \text{grad div } E_i.$$

Taking a solution similar to the isotropic case,

$$E_1 = a' e^{i2\pi(\alpha t - l_j x_j)/\lambda}, \text{ etc.}$$

and substituting these values for E_1, E_2 and E_3 in the equations for E_i, we get, the axes being rectangular

$$\frac{K'\alpha^2 E_1}{c^2} = E_1 - l_1(l_jE_j),$$

$$\frac{K''\alpha^2 E_2}{c^2} = E_2 - l_2(l_jE_j),$$

$$\frac{K'''\alpha^2 E_3}{c^2} = E_3 - l_3(l_jE_j).$$

If these three equations are simultaneously true then,

$$\frac{l_1^2}{K'\dfrac{\alpha^2}{c^2} - 1} + \frac{l_2^2}{K''\dfrac{\alpha^2}{c^2} - 1} + \frac{l_3^2}{K'''\dfrac{\alpha^2}{c^2} - 1} + 1 = 0$$

i.e., for every value of l_i there are in general two values of α so that two plane waves progress along the same wave normal with velocities α and α', Putting $c^2/K' = v'^2$, etc. where v' will be the velocity along $0x_1$, and rewriting the equation we get,

$$\frac{l_1^2}{\alpha^2 - v'^2} + \frac{l_2^2}{\alpha^2 - v''^2} + \frac{l_3^2}{\alpha^2 - v'''^2} = 0.$$

Or, in terms of the refractive index,

$$\frac{l_1^2}{\frac{1}{n^2}-\frac{1}{K'}}+\frac{l_2^2}{\frac{1}{n^2}-\frac{1}{K''}}+\frac{l_3^2}{\frac{1}{n^2}-\frac{1}{K'''}}=0.$$

Putting $x_1=l_1\alpha$, etc. we get,

$$\frac{x_1^2}{\alpha^2-v'^2}+\frac{x_2^2}{\alpha^2-v''^2}+\frac{x_3^2}{\alpha^2-v'''^2}=0.$$

These equations define the wave-normal surface.

6. The Relations between E_i, D_i, H_i, and the Wave-normal l_i in a Crystalline Dielectric

The field equations are,

$$\text{curl } H_i-\frac{\dot{D}_i}{c}=0$$

$$\text{curl } E_i+\frac{\dot{H}_i}{c}=0.$$

E_i and H_i are each proportional to $e^{i2\pi(\alpha t-l_jx_j)/\lambda}$ so that,

$$\dot{D}_j=i\frac{2\pi\alpha}{\lambda}D_j, \qquad \text{curl } E_j=i\frac{2\pi}{\lambda}[l, E]_j$$

$$\dot{H}_j=i\frac{2\pi\alpha}{\lambda}H_j, \qquad \text{curl } H_j=i\frac{2\pi}{\lambda}[l, H]_j.$$

Hence

$$i\frac{2\pi}{\lambda}[l, H]_j=i\frac{\alpha 2\pi}{c\lambda}D_j \quad \text{or} \quad \frac{\alpha}{c}D_j=[l, H]_j$$

and

$$i\frac{2\pi}{\lambda}[l, E]_j=-i\frac{\alpha 2\pi}{c\lambda}H_j \quad \text{or} \quad \frac{\alpha}{c}H_j=-[l, E]_j$$

i.e., D_i and H_i are at right-angles to l_i, and E_i lies in the plane of D_i and l_i but, generally, cannot be at right-angles to l_i since it does not coincide with D_i.

Eliminating H_i, we get,

$$\frac{\alpha D_i}{c}=-\frac{c}{\alpha}[l, [l, E]]_i$$

$$=-\{E_i-l_i(l_jE_j)\}\frac{c}{\alpha}$$

so that

$$D_i=\{E_i-l_i(l_jE_j)\}\frac{c^2}{\alpha^2}\cdot$$

This expression is the general one connecting D_i and E_i and in the isotropic case degenerates into $D_i = n^2 E_i$.

It can be written,

$$D_1 = \frac{c^2}{\alpha^2}\left\{\frac{D_1}{K'} - l_1(l_jE_j)\right\} = n^2\left\{\frac{D_1}{K'} - l_1(l_jE_j)\right\},$$

or

$$D_1 = -\frac{l_1(l_jE_j)}{\dfrac{1}{n^2} - \dfrac{1}{K'}}$$

where n is the refractive index in the direction l_i, and two similar equations for D_2 and D_3.

7. Relation between D_i and the Wave-normal l_i.

(a) Let the direction-cosines of D_i be m_i and its amplitude A. $E_1 = D_1/K'$, etc. so that in $E_1 = a'e^{i(2\pi/\lambda)(\alpha t - l_j x_j)}$, etc.

$$a' = A\frac{m_1}{K'}, \quad \text{etc.}$$

We have,

$$D_i\frac{\alpha^2}{c^2} = E_i - l_i(l_jE_j)$$

so that,

$$E_1 K'\frac{\alpha^2}{c^2} = E_1 - l_1(l_jE_j)$$

and two similar equations.

Hence,

$$m_1\left(\frac{c^2}{K'} - \alpha^2\right) = c^2 l_1\left(\frac{l_1 m_1}{K'} + \text{etc.}\right) = l_1 P \text{ say}$$

so that,

$$m_1 = \frac{l_1 P}{\dfrac{c^2}{K'} - \alpha^2}, \qquad m_2 = \text{etc.}$$

For every value of l_i there are in general two values of α and α', and, thus,

$$m_1 : m_2 : m_3 : = \frac{l_1}{\dfrac{c^2}{K'} - \alpha^2} : \frac{l_2}{\dfrac{c^2}{K''} - \alpha^2} : \frac{l_3}{\dfrac{c^2}{K'''} - \alpha^2}$$

$$m_1' : m_2' : m_3' = \frac{l_1}{\dfrac{c^2}{K'} - \alpha'^2} : \frac{l_2}{\dfrac{c^2}{K''} - \alpha'^2} : \frac{l_3}{\dfrac{c^2}{K'''} - \alpha'^2}.$$

Hence $m_i m_i'$ is proportional to,

$$\frac{l_1^2}{\left(\frac{c^2}{K'} - \alpha^2\right)\left(\frac{c^2}{K'} - \alpha'^2\right)} + \text{etc.}$$

$$= \frac{1}{\alpha^2 - \alpha'^2}\left\{\frac{l_1^2}{\frac{c^2}{K'} - \alpha^2} + \text{etc.} - \left(\frac{l_1^2}{\frac{c^2}{K'} - \alpha'^2} + \text{etc.}\right)\right\} = 0$$

so that m_i and m_i' are at right-angles, i.e., the two displacements for the wave-normal l_i are at right-angles.

(b) We have from (5),

$$\frac{l_1^2}{\frac{c^2}{K'} - \alpha^2} + \frac{l_2^2}{\frac{c^2}{K''} - \alpha^2} + \frac{l_3^2}{\frac{c^2}{K'''} - \alpha^2} = 0$$

and thus from (a),

$$\left(\frac{c^2}{K'} - \alpha^2\right) m_1^2 + \left(\frac{c^2}{K''} - \alpha^2\right) m_2^2 + \left(\frac{c^2}{K'''} - \alpha^2\right) m_3^2 = 0$$

so that,

$$\alpha^2 = \frac{c^2 m_1^2}{K'} + \frac{c^2 m_2^2}{K''} + \frac{c^2 m_3^3}{K'''}$$

and there is thus only one wave velocity corresponding to each direction of the displacement vector.

(c) From (a),

$$m_1 = \frac{l_1 g}{v'^2 - \alpha^2},$$

where

$$g^2 = \frac{1}{\sum \frac{l_1^2}{(v'^2 - \alpha^2)^2}}$$

and two similar equations for m_2, m_3.

8. Energy of Electrical and Magnetic Vibrations

We have,

$$u = \frac{E_i D_i}{8\pi} = \frac{c^2\{E_i^2 - (E_i l_i)^2\}}{8\pi\alpha^2}$$

and

$$t = \frac{H_i^2}{8\pi} = \frac{c^2[E, l]_i^2}{8\pi\alpha^2} = \frac{c^2\{E_i^2 l_i^2 - (E_i l_i)^2\}}{8\pi\alpha^2}$$

so that,

$$w = u + t = 2u = 2t = \frac{n^2\{E_i^2 - (E_il_i)^2\}}{4\pi}.$$

9. Relations between E_i, D_i, l_i, w and n

By (6) we have

$$D_i = n^2\{E_i - l_i(E_jl_j)\}$$

so that,

$$D_i^2 = n^4\{E_i^2 - (E_jl_j)^2\} = 4\pi n^2 w$$

and

$$n^2 = \frac{D_i^2}{4\pi w} = \frac{D_i^2}{E_iD_i}.$$

Thus

$$l_i = \frac{E_i - \dfrac{D_i}{n^2}}{E_jl_j} = \frac{E_i - \dfrac{D_i}{n^2}}{\sqrt{E_j^2 - \dfrac{D_j^2}{n^4}}}$$

$$= \frac{E_i - \dfrac{(E_jD_j)D_i}{D_j^2}}{\sqrt{E_j^2 - \dfrac{(E_jD_j)^2}{D_j^2}}} = \frac{E_iD_j^2 - D_i(E_jD_j)}{\sqrt{D_j^2\{E_j^2D_j^2 - (E_jD_j)^2\}}}.$$

10. The Energy Flow N_i in a Crystalline Dielectric—the Ray

We have from (3), $N_i = (c/4\pi)[E, H]_i$ and hence the energy flow diverges from the wave-normal at an angle θ say. This path of flow of the energy is called the ray and is at right-angles to E_i and H_i, and lies in the plane of l_i, E_i, and D_i. Inserting the values for H_i we get,

$$N_i = \frac{cn}{4\pi}[E, [E, l]]_i = \frac{cn}{4\pi}\{l_iE_j^2 - E_i(E_jl_j)\}.$$

Hence,

$$N_i^2 = \frac{c^2n^2}{(4\pi)^2}\{(E_j^2)^2 - E_i^2(E_jl_j)^2\}$$
$$= \frac{c^2n^2}{(4\pi)^2}E_i^2\{E_j^2 - (E_jl_j)^2\} = \frac{c^2E_i^2w}{4\pi}.$$

Also,

$$|N_i| \cos\theta = N_il_i = \frac{cn}{4\pi}l_i\{l_iE_j^2 - E_i(E_jl_j)\} = \frac{cw}{n} \qquad \text{by (8).}$$

Again,

$$
\begin{aligned}
N_iD_i &= \frac{cn}{4\pi}\{l_iE_j^2 - E_i(E_jl_j)\}D_i \\
&= \frac{cn^3}{4\pi}\{l_iE_j^2 - E_i(E_jl_j)\}\{E_i - l_i(E_jl_j)\} && \text{by (6)} \\
&= -\frac{cn^3}{4\pi}\{E_j^2 - (E_jl_j)^2\}E_il_i \\
&= -ncE_jl_jw && \text{by (8).}
\end{aligned}
$$

11. The Ray Index, r

N_i is the amount of energy crossing unit surface normally, in unit time. Imagine a cylinder erected on this base of unit area, with its length parallel to p_i, the direction of N_i, and of height β where β is the velocity along p_i. If the energy density within it is w then in unit time an amount of energy βw will pass through the unit area and thus,

$$|N_i| = \beta w.$$

We have,

$$N_il_i = \frac{cw}{n}.$$

So that

$$\beta = \frac{c}{n(p_il_i)} = \frac{\alpha}{p_il_i}$$

i.e., the wave velocity is the projection of the ray velocity on the wave-normal.

Also, since

$$N_i^2 = \frac{c^2E_i^2w}{4\pi}$$

$$\beta^2 = \frac{N_i^2}{w^2} = \frac{c^2}{4\pi w}E_i^2.$$

We define the ray index, r, by $r = c/\beta$ so that,

$$r^2 = \left(\frac{c}{\beta}\right)^2 = \frac{c^2 4\pi w}{c^2E_i^2} = \frac{E_iD_i}{E_i^2}$$

$$\left(\text{cf. } n^2 = \frac{D_i^2}{E_iD_i}\right).$$

Again, inserting the value for l_i in the expression for N_i, we get,

$$
\begin{aligned}
N_i &= \frac{cn(E_jD_j\{E_i(E_jD_j) - D_iE_j^2\}}{4\pi\sqrt{D_j^2\{E_j^2D_j^2 - (E_jD_j)^2\}}} \\
&= \frac{c\sqrt{E_jD_j}\,\{E_i(E_jD_j) - D_iE_j^2\}}{4\pi\sqrt{E_j^2D_j^2 - (E_jD_j)^2}}.
\end{aligned}
$$

Since $p_i = N_i/|N_i|$ and as,

$$|N_i| = \frac{c}{4\pi}\sqrt{(E_jD_j)E_j^2}$$

$$-p_i = \frac{D_iE_j^2 - E_i(E_jD_j)}{\sqrt{E_j^2\{E_j^2D_j^2 - (E_jD_j)^2\}}}$$

$$\left(\text{cf. } l_i = \frac{E_iD_j^2 - D_i(E_jD_j)}{\sqrt{D_j^2\{E_j^2D_j^2 - (E_jD_j)^2\}}}\right).$$

12. Relations of Wave and Ray Vectors

We have,

$$E_ip_i = 0; \qquad D_il_i = 0; \qquad p_il_i = \cos\theta; \qquad r = n\cos\theta$$

where θ is the angle between the wave-normal and the ray.

Hence in the equation

$$D_i = n^2\{E_i - l_i(E_jl_j)\} \qquad \text{of (6)}$$

$$D_ip_i = n^2\{E_ip_i - p_il_i(E_il_i)\} = -n^2(E_il_i)\cos\theta$$

or

$$E_il_i = -\frac{D_ip_i}{n^2\cos\theta}.$$

Since E_i, D_i and p_i are coplanar,

$$p_i = aD_i + bE_i$$

where a and b are constants.

Hence

$$p_i^2 = ap_iD_i + bp_iE_i \quad \text{i.e.,} \quad ap_iD_i = 1$$

and

$$p_il_i = al_iD_i + bl_iE_i \quad \text{i.e.,} \quad bl_iE_i = \cos\theta.$$

So that,

$$p_i = \frac{D_i}{p_jD_j} + \frac{E_i\cos\theta}{l_jE_j}$$

and

$$E_i = \frac{l_jE_j}{\cos\theta}\left\{p_i - \frac{D_i}{p_jD_j}\right\}.$$

Inserting the value for E_jl_j we get,

$$E_i = \frac{-D_jp_j\left(p_i - \dfrac{D_i}{p_jD_j}\right)}{n^2\cos^2\theta} = \frac{D_i - p_i(D_jp_j)}{r^2}.$$

Taking all these wave and ray equations together and comparing them, we see that they have the following correspondence in their terms;

$$\begin{array}{llllllllllllll} E_i & D_i & l_i & p_i & \alpha & n & K' & K'' & K''' & v' & v'' & v''' & c & \text{wave equations} \\ D_i & E_i & -p_i & -l_i & \frac{1}{\beta} & \frac{1}{r} & \frac{1}{K'} & \frac{1}{K''} & \frac{1}{K'''} & \frac{1}{v'} & \frac{1}{v''} & \frac{1}{v'''} & \frac{1}{c} & \text{ray equations} \end{array}$$

13. The Ray Equations

We have for the wave, from (6),

$$D_1 = \frac{-l_1(l_jE_j)}{\frac{1}{n^2} - \frac{1}{K'}} = \frac{-c^2l_1(l_jE_j)}{\alpha^2 - v'^2}$$

and two similar equations for D_2 and D_3 and also, from (5),

$$\frac{l_1^2}{\alpha^2 - v'^2} + \frac{l_2^2}{\alpha^2 - v''^2} + \frac{l_3^2}{\alpha^2 - v'''^2} = 0.$$

Substituting the corresponding terms from (12) we get,

$$E_1 = \frac{-p_1(p_jD_j)}{c^2\left(\frac{1}{\beta^2} - \frac{1}{v'^2}\right)}$$

and two similar equations for E_2 and E_3.

Also,

$$\frac{p_1^2}{\frac{1}{\beta^2} - \frac{1}{v'^2}} + \frac{p_2^2}{\frac{1}{\beta^2} - \frac{1}{v''^2}} + \frac{p_3^2}{\frac{1}{\beta^2} - \frac{1}{v'''^2}} = 0$$

and

$$\frac{p_1^2}{r^2 - K'} + \frac{p_2^2}{r^2 - K''} + \frac{p_3^2}{r^2 - K'''} = 0.$$

These equations define the ray surface which, if we put $x_i = \beta p_i$, may be written,

$$\frac{v'^2x_1^2}{v'^2 - \beta^2} + \frac{v''^2x_2^2}{v''^2 - \beta^2} + \frac{v'''^2x_3^2}{v'''^2 - \beta^2} = 0.$$

Thus to every value of p_i there are two values of β or r.

14. Relations of l_i and p_i

In any actual case only one of the vectors l_i and p_i is given and the other must be calculated from it. Knowing the l_i or p_i, the D_i and E_i can be calculated from the equations of (6) and (13).

(a) We have,

$$D_1 = E_1K' = \frac{-p_1(D_jp_j)\beta^2}{v'^2 - \beta^2}$$

and two similar equations for D_2 and D_3.

Also,

$$D_j p_j = -(E_j l_j) n^2 \cos\theta = \frac{c^2(E_j l_j)}{\alpha\beta}.$$

Thus,

$$\frac{l_1 \alpha}{\alpha^2 - v'^2} = \frac{p_1 \beta}{\beta^2 - v'^2}$$

and two similar equations for l_2, l_3, p_2, p_3.

(b) We have already, from (5), the relations between the direction-cosines of the wave-normals so that we derive,

$$\frac{l_1 p_1}{\beta^2 - v'^2} + \frac{l_2 p_2}{\beta^2 - v''^2} + \frac{l_3 p_3}{\beta^2 - v'''^2} = 0.$$

(c) Again, from (a) we can write,

$$\beta p_1 - \alpha l_1 = \frac{l_1 \alpha(\beta^2 - \alpha^2)}{\alpha^2 - v'^2}$$

and two similar equations.

Squaring and adding these three equations, we have,

$$\alpha^2(\beta^2 - \alpha^2) = \frac{1}{\sum\left\{\dfrac{l_1}{\alpha^2 - v'^2}\right\}^2} = g^2 \qquad \text{by (7)}$$

α is already known by the wave equation in terms of l_i and therefore $\alpha^2(\beta^2-\alpha^2)$, and thus β, can be expressed in terms of l_i. Hence p_i is expressed as a function of l_i.

We can write,

$$p_1 = \frac{l_1}{\alpha\beta}\left\{\alpha^2 + \frac{g^2}{\alpha^2 - v'^2}\right\}$$

and two similar equations for p_2, p_3, l_2, l_3.

Hence, since to each value of l_i there are two values of α, there must in general be two rays p_i for each l_i.

(d) By the relations of (12) we can write,

$$l_1 = \alpha\beta p_1 \left\{\frac{1}{\beta^2} + \frac{g^2}{\dfrac{1}{\beta^2} - \dfrac{1}{v'^2}}\right\}$$

and two similar equations for l_2, l_3, p_2, p_3.

Hence, since to each value of p_i there are two values of β, there must in general be two wave-normals l_i for each p_i.

15. Relations of the Wave-normal Surface and the Ray Surface

A small change in either the electrical intensity or the displacement will bring about a small change in the l_i relative to the p_i. We now investigate this relation.

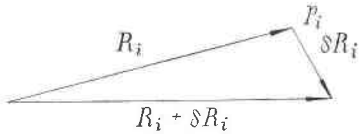

FIG. 1. The direction of the tangent plane to the ray surface.

Let p_i be the ray direction and β the ray velocity in this direction. If $R_i=\beta p_i$ then the end of R_i sweeps out the ray surface. Let l_i be the wave-normal associated with p_i according to (14).

Since $p_i c=rR_i$, we have from (12),

$$c^2E_i = (R_i^2)D_i - R_i(D_jR_j).$$

If the quantities be considered as functions of a variable t, say, then the displacement of R_i will be $\delta R_i=\dot{R}_i\delta t$ where the dot indicates differentiation with respect to t.

We have,

$$c^2\dot{E}_i = \dot{D}_i(R_j^2) + 2D_i(R_j\dot{R}_j) - \dot{R}_i(D_jR_j) - R_i(D_j\dot{R}_j) - R_i(\dot{D}_jR_j).$$

Multiplying these equations by the appropriate D_i and adding, we have,

$$\begin{aligned} c^2\dot{E}_iD_i &= (\dot{D}_iD_i)(R_i^2) - (D_iR_i)(\dot{D}_iR_i) + 2\{(D_i^2)(\dot{R}_iR_i) - (R_iD_i)(\dot{R}_iD_i)\} \\ &= \dot{D}_i\{(R_i^2)D_j - R_j(D_iR_i)\}_i + 2\dot{R}_i\{(D_i^2)R_j - D_j(R_iD_i)\}_i \\ &= c^2\dot{D}_iE_i + 2\dot{R}_i[[D, R], D]_i \end{aligned}$$

i.e., $2\ \dot{R}_i[[D, R], D]_i=0$ since $D_i\dot{E}_i=E_i\dot{D}_i$.

The vector $[[D, R], D]_i$ is perpendicular to the normal to the plane of D_i and p_i and is also at right-angles to D_i. It is therefore parallel to l_i. We have then $\dot{R}_i[[D, R], D]_i=0$ and hence the displacement of R_i, being $\delta R_i=\dot{R}_i\delta t$, must be at right-angles to l_i since $l_i\delta R_i=0$. That is, the tangent plane to the ray surface at the end of a radius-vector is always at right-angles to the corresponding wave-normal.

The principal axes of the wave-normal surface and the ray surface coincide and therefore the wave-normal surface is the pedal surface of the ray surface and conversely the ray surface is the envelope of the planes at right-angles to the radii-vectores of the wave-normal surface.

16. Derivation of the Wave-normal Ellipsoid[4]

By (5) we have for the relation between the wave-normal l_i and the

[4] This surface is also called the indicatrix (Fletcher), the indexellipsoid (Pockels and Szivessy), the normalenellipsoid (Born). As the term "indicatrix" has long had a definite

velocities α of the two waves propagated along it,

$$\frac{l_1^2}{\alpha^2 - v'^2} + \frac{l_2^2}{\alpha^2 - v''^2} + \frac{l_3^2}{\alpha^2 - v'''^2} = 0.$$

Hence if waves travel outwards from a point within the crystal in all directions, the limits of their travel after unit time along the normals will form a twofold surface, the wave-normal surface, which is of the fourth degree. Such a surface is a complicated one and it is more convenient to take as reference an ellipsoid derived from the energy equation (4),

$$\frac{D_1^2}{K'} + \frac{D_2^2}{K''} + \frac{D_3^2}{K'''} = 8\pi u.$$

Taking the x_i as measures of the D_i and with suitable adjustments we can put,

$$\frac{x_1^2}{K'} + \frac{x_2^2}{K''} + \frac{x_3^2}{K'''} = 1, \quad \text{or} \quad \frac{x_1^2}{n'^2} + \frac{x_2^2}{n''^2} + \frac{x_3^2}{n'''^2} = 1.$$

This is an ellipsoid whose principal axes coincide with the dielectric axes and are proportional to the roots of the principal dielectric constants or to the principal refractive indices. It is called here the wave-normal ellipsoid. By reference to it the course of the propagation of light in crystals can be illustrated and examined in the following manner.

17. Refractive Indices for the Wave-normal l_i

Let a radius-vector of the ellipsoid represent a wave-normal l_i. Then, by (6), D_i, which we shall take as the "vibration," must lie in a plane at right-angles to this radius vector. Let $l_i x_i = 0$ be such a plane through the origin. It will cut the ellipsoid in an ellipse and the principal axes of the ellipse give in direction and magnitude the two values of D_i demanded by electromagnetic theory. That this is so we prove as follows.

For the radius-vector of length r,

$$r^2 = x_i^2 = f(x_i) \quad \text{say}.$$

We have therefore to find the maximum and minimum values for r having regard to the conditions,

$$0 = \frac{x_1^2}{K'} + \frac{x_2^2}{K''} + \frac{x_3^2}{K'''} - 1 = \phi(x_i) \quad \text{say},$$

and

$$0 = l_i x_i = \psi(x_i) \quad \text{say}.$$

---

meaning in the geometry of higher surfaces and curves and as the expression "indexellipsoid" is ambiguous, Born's name is probably best. "Wave-normal ellipsoid" is used here to make the reference as precise as possible.

Forming $df+\lambda d\phi+2\,\mu d\psi$ where λ and $2\,\mu$ are undetermined multipliers, and equating the coefficients of the dx_i to zero,

$$x_1+\frac{\lambda x_1}{K'}+\mu l_1=0,\qquad x_2+\frac{\lambda x_2}{K''}+\mu l_2=0,\qquad x_3+\frac{\lambda x_3}{K'''}+\mu l_3=0.$$

The values of x_i which satisfy these equations are those that determine the turning values of r^2. Multiplying the equations by the x_i and adding gives $r^2=-\lambda$ and this gives on substitution,

$$x_1=\frac{\mu l_1}{\dfrac{r^2}{K'}-1},\qquad x_2=\frac{\mu l_2}{\dfrac{r^2}{K''}-1},\qquad x_3=\frac{\mu l_3}{\dfrac{r^2}{K'''}-1}.$$

Inserting these values in $l_ix_i=0$,

$$\frac{l_1^2}{\dfrac{r^2}{K'}-1}+\frac{l_2^2}{\dfrac{r^2}{K''}-1}+\frac{l_3^2}{\dfrac{r^2}{K'''}-1}=0$$

which gives two solutions for r^2, the turning values.

By (5) this is the equation which defines the refractive indices of the two waves proceeding along the wave-normal l_i so that the lengths of the major and minor axes of the elliptic section give the refractive indices of the waves propagated along the radius-vector l_i.

Again, if we multiply the equations by the l_i and add, we get,

$$\lambda\left(\frac{x_1l_1}{K'}+\frac{x_2l_2}{K''}+\frac{x_3l_3}{K'''}\right)+\mu=0$$

which gives then, as the three equations defining the x_i for the turning values of r^2,

$$x_1-\frac{r^2x_1}{K'}+l_1r^2\left(\frac{x_1l_1}{K'}+\frac{x_2l_2}{K''}+\frac{x_3l_3}{K'''}\right)=0$$

and two similar equations. If in these equations the x_i are replaced by the D_i and x_1/K', etc. by E_1, etc., then,

$$D_1=n^2\left(\frac{D_1}{K'}-l_1(E_il_i)\right)$$

and two similar equations, and these by (6) define the electrical displacements associated with the wave-normal l_i.

18. Vibration Directions for the Wave-normal l_i

From the values obtained in (17) for the x_i the ratios of the direction-cosines of the axes of the elliptic section at right-angles to l_i are,

$$\frac{l_1}{v'^2-\alpha^2}:\frac{l_2}{v''^2-\alpha^2}:\frac{l_3}{v'''^2-\alpha^2}$$

which are the values determined for the direction-cosines m_i of D_i in (7). We note further that these electrical displacements are at right-angles as required by (7).

19. Ray Direction and Ray Index for the Wave-normal l_i

Let x_i' be the end of one of the principal axes of the elliptic section at

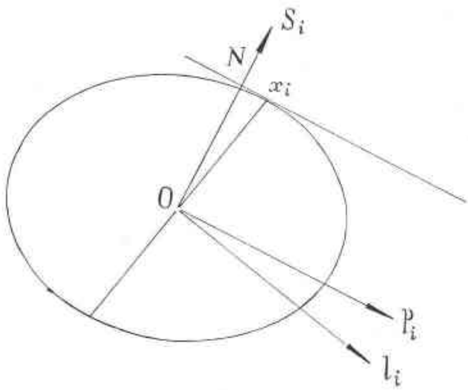

FIG. 2. Section of the wave-normal ellipsoid containing the wave-normal l_i and one of the principal axes Ox_i of the elliptic section at right-angles to it.

right-angles to the wave-normal l_i. This axis defines one value of D_i in magnitude and direction, i.e., on the appropriate scale, $x_i' = D_i$. By (6),

$$D_1 = \frac{l_1(l_iE_i)}{\dfrac{1}{n'^2} - \dfrac{1}{n^2}}$$

so that,

$$n = |D_i| = \left[\sum \left\{ \frac{l_1(l_iE_i)}{\dfrac{1}{n'^2} - \dfrac{1}{n^2}} \right\}^2 \right]^{1/2}.$$

By (14)

$$g^2 = \frac{1}{\sum \left(\dfrac{l_1}{\alpha^2 - v'^2} \right)^2} = \frac{c^4}{\sum \dfrac{l_1^2}{\left(\dfrac{1}{n'^2} - \dfrac{1}{n^2} \right)^2}}.$$

Hence,

$$l_iE_i = \frac{ng}{c^2}.$$

We have thus for the co-ordinates x_i',

$$x_1' = \frac{l_1 ng}{c^2\left(\dfrac{1}{n'^2} - \dfrac{1}{n^2}\right)}$$

and similar equations for x_2' and x_3'.

The tangent to the elliptic section at x_i' is at right-angles to Ox_i' and is thus perpendicular to the plane of l_i and Ox_i' since it also lies in the central plane at right-angles to l_i. Draw the tangent plane to the ellipsoid at x_i' and let ON, with direction-cosines s_i, be the normal to this plane from the origin. Since ON is also at right angles to the line through N parallel to the tangent to the elliptic section at x_i', it must be co-planar with l_i and Ox_i'.

We have,

$$s_1 = \frac{\dfrac{x_1'}{n'^2}}{\left\{\sum\left(\dfrac{x_1}{n'^2}\right)^2\right\}^{1/2}} = \frac{x_1'q}{n'^2}, \quad \text{say,}$$

$$= \frac{l_1 ngq}{c^2 n'^2\left(\dfrac{1}{n'^2} - \dfrac{1}{n^2}\right)}$$

and similar equations for s_2 and s_3.

By (14) we have, for the ray corresponding to the wave-normal l_i,

$$p_1 = \frac{rl_1}{n}\left\{1 - \frac{n^2g^2}{c^4\left(\dfrac{1}{n'^2} - \dfrac{1}{n^2}\right)}\right\}$$

and similar equations for p_2 and p_3.

Hence,

$$s_ip_i = \frac{gqr}{c^2}\sum\left[\frac{l_1^2}{n'^2\left(\dfrac{1}{n'^2} - \dfrac{1}{n^2}\right)}\left\{1 - \frac{n^2g^2}{c^4\left(\dfrac{1}{n'^2} - \dfrac{1}{n^2}\right)}\right\}\right]$$

$$= \frac{gqr}{c^2}\left\{\sum\frac{l_1^2}{n'^2\left(\dfrac{1}{n'^2} - \dfrac{1}{n^2}\right)} - \frac{g^2}{c^4}\sum\frac{l_1^2n^2}{n'^2\left(\dfrac{1}{n'^2} - \dfrac{1}{n^2}\right)^2}\right\}.$$

From (5)

$$\sum\frac{l_1^2}{n'^2\left(\dfrac{1}{n'^2} - \dfrac{1}{n^2}\right)} = 1$$

also,

$$\sum\frac{l_1^2}{\left(\dfrac{1}{n'^2} - \dfrac{1}{n^2}\right)} = 0$$

so that,

$$\sum \frac{l_1^2}{n'^2\left(\frac{1}{n'^2}-\frac{1}{n^2}\right)^2} = \sum \frac{l_1^2}{n^2\left(\frac{1}{n'^2}-\frac{1}{n^2}\right)^2}$$

$$\therefore \quad s_i p_i = 0$$

i.e. p_i is given by the two conditions, (a), it lies in the plane of l_i, Ox_i' and ON, and, (b), it is at right-angles to ON.

Further, $ON = n \cos\theta$ and hence is equal to the ray index r. We note also that, from the geometry of the ellipsoid, ON will be a principal semi-diameter of the cross section of the cylinder with axis p_i and tangential to the ellipsoid along its intersection with the plane diametral to p_i.

These relations being established between the wave-normal ellipsoid and the electromagnetic vectors, all of the remaining relations depend only on the geometry of the ellipsoid itself. Beer, Becke and Wright[5] have pointed out that the full optical relations can be developed from the wave-normal ellipsoid by consideration of the cones defined by the intersection of the ellipsoid with a sphere of variable radius corresponding to the refrractive index. In concluding this note it is worth drawing attention to the role played in this matter by the focal lines of these cones.

If the wave-normal ellipsoid is

$$\sum \frac{x_1^2}{n'^2} = 0$$

the radii-vectors of length corresponding to the variable refractive index r give the "equivibration" cone

$$\sum x_1^2\left(\frac{1}{n'^2}-\frac{1}{r^2}\right) = 0$$

and this degenerates into the two planes of cyclic section of the ellipsoid for $r = n''$. Since the coefficients of x_1^2 etc. in the equations to the cone and the ellipsoid differ only by a constant term, the directions of the circular sections are the same in each. The real focal lines of these equivibration cones are given by,

$$x_2^2\frac{(r^2-n''^2)}{(n''^2-n'^2)} - x_3^2\frac{(n'''^2-r^2)}{(n'''^2-n'^2)} = 0 \qquad \text{for } n'' < r < n'''$$

and

$$x_1^2\frac{(r^2-n'^2)}{(n'''^2-n'^2)} - x_2^2\frac{(n''^2-r^2)}{(n'''^2-n''^2)} = 0 \qquad \text{for } n' < r < n''$$

[5] Beer, A.: Grunert's Arch., Th. 16, 223–229 (1851).
Becke, F.: *Tschermak's Min. u. Pet. Mitt.*, **24,** 1–34 (1905).
Wright, F. E.: *Jour. Opt. Soc. Am.*, **7,** 779–817 (1923).

i.e. the cones fall into two sets, the cyclic sections of the ellipsoid forming the boundary between them.

We note in passing that a property of the focal lines of such cones is that the section of the cone by any plane at right-angles to one of them is a conic having for focus the point where the focal line meets the plane. Further, the directrix of the cone corresponding to the point on the focal line lies in the plane of section and is at right-angles to the plane of the focal lines. It is also the directrix of the conic section.

A tangent plane to the equivibration cone along a generator, l_i, say, is

$$\sum l_1 \left(\frac{1}{n'^2} - \frac{1}{r^2}\right) x_1 = 0.$$

It is a diametral plane of the ellipsoid and the generator, being a radius of the sphere and thus at right-angles to the curve of its intersection with the ellipsoid, is a principal axis of the elliptic section. Hence the normals to the tangent planes of the equivibration cone give the directions of propagation for which one refractive index is r. They form the cone

$$\sum \frac{x_1^2}{\left(\frac{1}{n'^2} - \frac{1}{r^2}\right)} = 0,$$

the equirefringence cone, which is reciprocal to the equivibration cone. The relations of two such reciprocal cones are shown in Fig. 3. The focal lines of each cone are at right-angles to the circular sections of the other and the normal plane common to the two cones namely,

$$\sum \frac{\left(\frac{1}{n''^2} - \frac{1}{n'''^2}\right)}{l_1} x_1 = 0,$$

bisects the angle between the planes through the generator and the focal lines in each case. The direction-cosines of the generator of the equirefringence cone corresponding to the vibration direction l_i are in the ratio,

$$l_1\left\{\left(\frac{1}{n'''^2} - \frac{1}{n'^2}\right) l_3^2 - \left(\frac{1}{n'^2} - \frac{1}{n''^2}\right) l_2^2\right\}$$
$$:l_2\left\{\left(\frac{1}{n'^2} - \frac{1}{n''^2}\right) l_1^2 - \left(\frac{1}{n''^2} - \frac{1}{n'''^2}\right) l_3^2\right\}$$
$$:l_3\left\{\left(\frac{1}{n''^2} - \frac{1}{n'''^2}\right) l_2^2 - \left(\frac{1}{n'''^2} - \frac{1}{n'^2}\right) l_1^2\right\}.$$

The focal lines of the equirefringence cones are therefore at right-angles to the cyclic planes of the equivibration cones and of the ellipsoid and are thus the same for all. They constitute the wave-normal axes and the

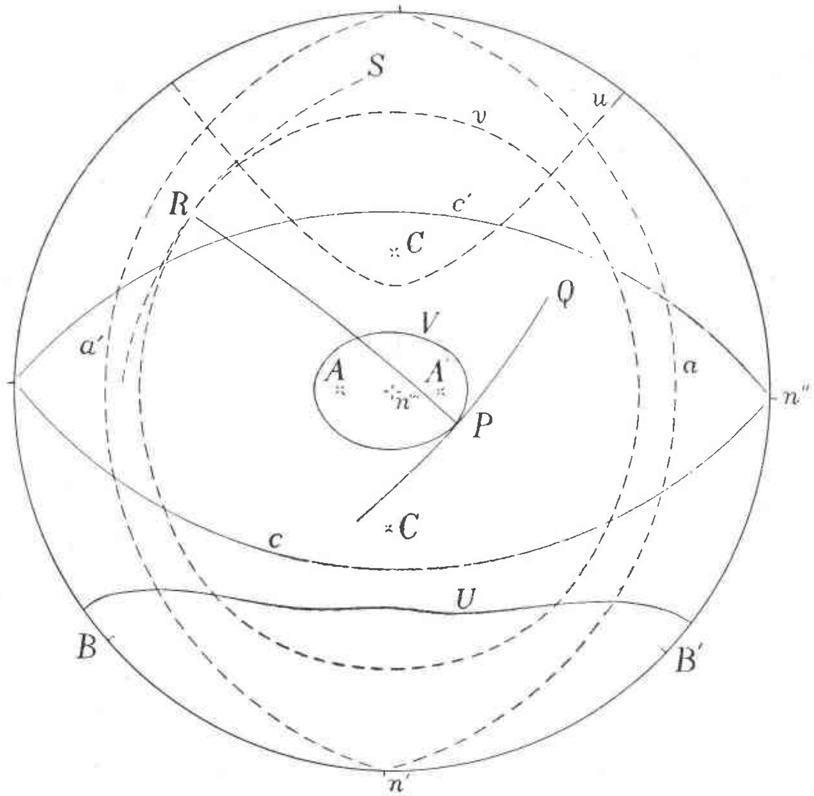

FIG. 3. Reciprocal cones in stereographic projection. V is a cone of the set $n''<r<n'''$ with focal lines at A and A', and U, focal lines at B and B', one of the set $n'<r<n''$. c and c' are the planes of circular section, $r=n''$. The cone reciprocal to V is v and its focal lines are C and C' at right-angles to the cyclic sections c and c'. The normal plane common to the two cones is RP and it bisects, in each cone, the angle between the planes through the generator and the focal lines. The circular sections of v are a and a' at right-angles to the focal lines A and A' of V. RS is the tangent plane to v at right angles to OP and PQ the tangent plane to V at right-angles to OR.

refractive indices for the crystal are thus given by cones set about these optic axes. These fall into two sets corresponding to the two sets of equivibration cones as shown in Fig. 3, i.e., (i) for $n''<r<n'''$ the cones lie between the principal section $n'n''$ and the wave-normal axes. Their circular sections, being at right-angles to the focal lines of the reciprocal, equivibration cones, are perpendicular to the plane $n''n'''$. (ii) for $n'<r<n''$ they lie between the principal section $n''n'''$ and the wave-normal axes, their circular sections being at right-angles to the plane of n' and n''. Since they have the same focal lines, these two sets of equire-

fringence cones intersect, thus determining the refractive indices for the direction of propagation defined by the common generators.

Since we are concerned only with directions, it is sufficient to consider the intersection of these equivibration and equirefringence cones with a sphere of reference in which they depict for the crystal the refractive indices and the directions of vibration.[6] Since for a second degree cone the sum of the angles between any generator and the focal lines is a constant, namely the angle between the generators in the plane of the focal lines, the curves of intersection of these cones with the sphere are analogous to plane ellipses.

The directions of vibration for a generator defining a propagation direction is given by the intersection of the normal plane through the generator with the plane at right-angles to the generator. This plane, as we have seen, bisects the angle between the planes containing the generator and the focal lines. If the direction-cosines of the generator are l_i the direction-cosines of the vibration direction are in the ratio,

$$\begin{aligned} l_1 &\left\{\left(\frac{1}{n''^2} - \frac{1}{r^2}\right)\left(\frac{1}{n'^2} - \frac{1}{n'''^2}\right) l_3^2 - \frac{1}{n'''^2} - \frac{1}{r^2}\left(\frac{1}{n''^2} - \frac{1}{n'^2}\right) l_2^2\right\} \\ &: l_2\left\{\left(\frac{1}{n'''^2} - \frac{1}{r^2}\right)\left(\frac{1}{n''^2} - \frac{1}{n'^2}\right) l_1^2 - \left(\frac{1}{n'^2} - \frac{1}{r^2}\right)\left(\frac{1}{n'''^2} - \frac{1}{n''^2}\right) l_3^2\right\} \\ &: l_3\left\{\left(\frac{1}{n'^2} - \frac{1}{r^2}\right)\left(\frac{1}{n'''^2} - \frac{1}{n''^2}\right) l_2^2 - \left(\frac{1}{n''^2} - \frac{1}{r^2}\right)\left(\frac{1}{n'^2} - \frac{1}{n'''^2}\right) l_1^2\right\} \end{aligned}$$

[6] See Johannsen, A.: Manual of Petrographic Methods, New York, (1918), pp. 429–434, and Wright, F. E.: *op. cit.*, pp. 790–792 for diagrams of projections of these curves of intersection on the principal planes. Both of these works give a summary of the development of this method of analysis of the optical properties of crystals.

Manuscript received Jan. 30, 1952